$13^{\textbf{th}}$ Annual Johns Hopkins Math Tournament
Saturday, February 2, 2013
Explorations Unlimited Round – Additive Number Theory

1. Introduction

Additive number theory has seen a spate of relatively recent results on very old, very hard, open problems; for example, the Green-Tao theorem settled a long-standing problem in 2004,[1] and the seminal but older partial results by Chen on both the Goldbach Hypothesis and the Twin Primes problems remain current today.[2,3] While these results are too complex to be summed up here, the older proofs presented here are worth revisiting to understand where the recent developments are coming from. We will work our way up to Dyson's Transform; this will give you all of the tools you will need to understand the proof of Mann's Theorem on your own.

2. The Basics: Core Definitions and Notation

The set of natural numbers, $\mathbb{N}$, is defined here as $\mathbb{N} = \{0, 1, 2, 3, ...\}$.
The set of real numbers is written as $\mathbb{R}$.
We use the notation $a \in A$ to denote that a is an element of a sequence called A.
We use the notation $|A|$ to denote the cardinality of a set A; that is, the number of elements in A.

3. Sequences

Additive number theory, at its core, is about working with sequences. Here, we introduce addition of sequences, and later on, we'll work with the Dyson transform. At the heart of it all is the realization that having the mathematical tools to say things about several sequences at once is very useful.

Definition 3.1 (Sequences)**.** A **sequence** is an ordered list of mathematical objects. Here, we are concerned with sequences of natural numbers, or of prime numbers, or of real numbers. A sequence, unlike a set, is allowed to have repeats; $S = \{1, 1, 1, 1\}$ is a sequence, but not a set. We denote a sequence by the capital letter S, and the n^{th} term in this sequence as a_n, and write $S = \{a_1, a_2, a_3, ...\}$. (so $n \geq 1$).

Definition 3.2 (Monotone Increasing/Decreasing Sequences)**.** Let S be a sequence, n and m positive integers. We say S is **monotone increasing** if, for every n, $a_n \leq a_{n+m}$ holds for all m.
Similarly, we say S is **monotone decreasing** if, for every n, $a_n \geq a_{n+m}$ holds for all m.
When $\leq$ can be replaced with $<$, we call this a **strictly increasing** sequence. When $\geq$ can be replaced with $>$, we call this a **strictly decreasing** sequence.

Problem 1 (Sequence Problem: Euler's Continued Fraction; 5 points)**.** Consider the monotone decreasing sequence given by $a_0 = 1$, $a_n = 1 + \frac{1}{1+a_{n-1}}$. As n becomes very large (goes to infinity), what happens to a_n? *You do not need calculus to solve this problem*: this is the same as solving for x in

$$1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}} = x$$

Definition 3.3 (Infimum of a sequence)**.** Let $S = \{a_1, a_2, a_3, ...\}$ be a sequence. Let $a, x \in \mathbb{R}$. We say that a is the **infimum** of S if $a \leq a_n \forall n$ and if $x \leq a_m\ \forall m$, $x \leq a$.

Problem 2 (Uniqueness of Infimum; 4 points)**.** Prove that if a sequence S has an infimum, then this infimum is unique.

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[1]Green, Ben; Tao, Terence (2008), "The Primes Contain Arbitrarily Long Arithmetic Progressions", Annals of Mathematics **167** (2): 481-547

[2]Jingrun, Chen (1978), "On The Representation of a Large Even Integer as the Sum of a Prime and the Product of at Most Two Primes. II.", Scientia Sinica **16**: 421-430.

[3]Melvyn B. Nathanson, *Additive Number Theory: The Classical Bases* (New York: Springer, 1996), 272.

Definition 3.4 (Sum of Sequences)**.** Consider k monotone increasing sequences which begin with zero, which we refer to as $S^{(1)}$, $S^{(2)}$,..., $S^{(k)}$:
$S^{(1)} = \{0, a_1^{(1)}, a_2^{(1)}, ..., a_m^{(1)}, ...\}$
$S^{(2)} = \{0, a_1^{(2)}, a_2^{(2)}, ..., a_m^{(2)}, ...\}$
...
$S^{(k)} = \{0, a_1^{(k)}, a_2^{(k)}, ..., a_m^{(k)}, ...\}$

Pick a number from each sequence in order from smallest to greatest k, and add these k numbers together. For example, $a_1^{(1)} + a_3^{(2)} + ... + a_2^{(k)}$. Now do this for every possible choice of numbers from each sequence. Once this is done, order the numbers from smallest to largest. $\{0, n_1, n_2, ..., n_m, ...\}$
We call this new sequence the sum of the given sequences $S^{(1)}$, $S^{(2)}$,..., $S^{(k)}$, and write this sum of sequences as:
$S = S^{(1)} + S^{(2)} + ... + S^{(k)} = \sum_{i=1}^{k} S^{(i)}$

Definition 3.5 (Special Case: Sums of Sets)**.** In the case where these sequences are sets, we call this operation the **sumset**. The sum of two sets is not necessarily a set; for example, $A = \{-1, 1\}, B = \{-1, 1\}, A + B = \{-2, 0, 0, 2\}$.

Definition 3.6 (Difference of Sets)**.** We define $A - B$ for sets A and B as taking the difference of the elements rather than adding them together.

Problem 3 (1 point)**.** Let $S^{(1)} = \{0, 1\}$; $S^{(2)} = \{0, 2\}$. Compute $S^{(1)} + S^{(2)}$.

Problem 4 (Sumset Lower Bound; 1 point)**.** Let A and B be non-empty finite sets such that $A + B$ is still a set. Prove that $|A + B| = |A||B|$.

4. Basis and Density of a Sequence

Now that we can manipulate sequences, we introduce two definitions- the basis and the density of a sequence- that will allow us to say useful things about complicated sequences by breaking them down into sums of simpler sequences that we know a lot about.

Definition 4.1 (Basis of order k)**.** We say a sequence S is a **basis of order** k if the sum of k identical sequences S contains all the natural numbers.

Definition 4.2 (Schnirelmann Density of Sequences)**.** Let $S = a_1 = 0, a_2, a_3, ..., a_n, ...$ be a strictly increasing sequence composed only of natural numbers.
Let $S(n)$ be the number of unique natural numbers in the sequence S in the interval $[1, n]$.
Form a new sequence $S' = \{\frac{S(1)}{1}, \frac{S(2)}{2}, \frac{S(3)}{3}, ... \frac{S(m)}{m}, ...\}$ for all n.
Denote the infimum of this sequence by $d(S)$. This is the Schnirelmann Density of the sequence S.

This deserves a bit of explanation. $\frac{S(n)}{n}$ is an average value of how dense a sequence is from 1 to n. To have a meaningful value that is not misled by early terms, we need to consider every possible n value; in the case of an infinite sequence, this is not always easy and would require notions of calculus, so we won't deal with it directly. The following problems should help build some familiarity with this definition.

Problem 5 (15 points)**.** Prove that $0 \leq \frac{S(n)}{n} \leq 1$ for all $n \neq 0$.

Problem 6 (5 points)**.** Let $mS = \underbrace{S + ... + S}_{\text{m times}}$. Prove that if S is a basis of order k and $0 \in S$, then mS is a basis for any $m \geq k$.

Problem 7 (Density and natural numbers; 10 points)**.** Prove that a sequence S contains every natural number if and only if $d(S) = 1$.

5. Several Results by Schnirelmann

Definition 5.1 (Schnirelmann's Inequality)**.** By itself, this last problem doesn't look like much. However, it becomes a powerful fact once we have a working method to find the density of infinite sequences. Schnirelmann proved just such a result. Let A and B be strictly increasing sequences composed only of natural numbers. Then $d(A+B) \geq d(A) + d(B) - d(A)d(B)$. We will now prove this.

Problem 8 (Schnirelmann's Inequality; 55 points total)**.** To streamline notation, let: $d(A) = \alpha, d(B) = \beta, A + B = C, d(C) = \gamma$, $l = a_{k+1} - a_k - 1$. Let $B(l)$ be the number of unique numbers of B in the segment $[1, l]$. If $l = 0$, let $B(l) = 0$. Let $A(n)$ be the number of unique natural numbers in A that appear in the segment $[1, n]$. The key to this proof is to break down the segment $[1, n]$ into two parts: elements in A, and segments between such elements (which are not in A).

- Prove that $C(n) \geq A(n) + \Sigma B(l)$, with $\Sigma B(l)$ meaning the sum of $B(l)$'s for every segment of natural numbers that are not in A that are between two elements in A. (Remember, we broke up the interval $[1, n]$ into different parts.) *[15 points]*
- Prove that $B(l) \geq \beta l$. *[5 points]*
- Use this to prove that $C(n) \geq A(n) + (n - A(n))\beta$. *[15 points]*
- Use a similar method from (2) and (3) to prove that: $C(n) \geq \alpha n(1-\beta) + n\beta$. *[5 points]*
- Use this to prove that $\frac{C(n)}{n} \geq \alpha + \beta - \alpha\beta$, which completes our proof. *[15 points]*

Problem 9 (Generalized Schnirelmann's Inequality; 25 points)**.** Note that we can rearrange Schnirelmann's Inequality as such: $1 - d(A+B) \leq (1 - d(A))\,(1 - d(B))$
Prove the case for an arbitrary number of summands: $1 - d(A_1 + A_2 + ... + A_k) \leq \Pi_{i=1}^{k}\,(1 - d(A_i))$

Problem 10 (Schnirelmann's Lemma; 25 points)**.** Prove that if $A(n) + B(n) > n - 1$, then n occurs in $A + B$.

Problem 11 (Schnirelmann's Theorem; 25 points)**.** It can be proven using calculus that if $d(A) > 0$, then $d(kA) > \frac{1}{2}$ for sufficiently large k.
Using the previous results and this given fact, prove that any sequence A with $d(A) > 0$ is a basis for the sequence of natural numbers.

6. Aside: historical note

Schnirelmann used the theorem now named after him, alongside his proof that P+P for P the sequence of prime numbers had positive density, to prove that all natural numbers could be expressed as the sum of sufficiently many primes. From this, we can say that any natural number greater than 1 can be expressed as the sum of at most k primes. How low can k be? Goldbach's Conjecture hypothesizes that for every *even* natural number, $k = 2$. The best known so far is that $k \leq 7$, a result due to Ramaré in 1995.[4]

7. Dyson's Transform: moving towards Mann's Theorem

We can actually improve on Schnirelmann's Inequality, and this is result is known as Mann's Theorem. The simplest way to obtain a proof of Mann's Theorem is to use something called the **Dyson Transform**, which we define here.

Definition 7.1 (Dyson's Transform)**.** Let A and B be two strictly increasing sequences; since they are strictly increasing, these are sets. Suppose $0 \in A \cap B$. Let $e \in A$. Then the **Dyson Transform** of these sequences is forming two new sequences as follows:
$A' = A \cup (B + \{e\})$
$B' = B \cap (A - \{e\})$

If this seems a bit arbitrary, perhaps the following four exercises on its basic properties will help build some intuition. It is a very useful way to work with two sets at the same time, and this transform shows up in many other proofs in additive number theory.

[4] Ramaré, Olivier (1995), "On Šnirel'man's Constant", Annali Della Scuola Normale Superiore di Pisa Classe di Scienze, **22**: 645-706

Problem 12 (Dyson Transform Property 1; 7 points)**.** Prove that $0 \in A' \cup B'$.

Problem 13 (Dyson Transform Property 2; 7 points)**.** Prove that $d(B') \leq d(B)$.

Problem 14 (Dyson Transform Property 4; 7 points)**.** Prove that $|A'| + |B'| = |A| + |B|$. (assume A and B are finite, so our definition of cardinality works)

Problem 15 (Dyson Transform Property 5; 7 points)**.** Prove that $A' + B' \subset A + B$.

Mann's Theorem

Mann's Theorem states that if $0 \in A \cap B$, then $d(A + B) \geq min\{1, d(A) + d(B)\}$.
The full proof is far too long to be asked as a problem here, but you now have all of the prerequisites to understand the proof. A good version can be found in Andrew Granville's "Additive Combinatorics" notes, which you can find here (page 9): `http://www.dms.umontreal.ca/~andrew/PDF/AddComb2010.pdf`.

Problem 16 (A piece of Mann's Theorem; 6 points)**.** Assume $\frac{A(m)+B(m)}{m} \geq 1$. Prove Mann's Theorem for this case.