JHMT 2013 Probability and Combinatorics Test
February 2, 2013

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

Problem 1. Spot, a cute little puppy, is playing a game with Frosty, the evil abominable snowman. Spot chooses a random perfect square greater than 1 but less than 10, while Frosty chooses a random odd number greater than 1 but less than 10. Spot wins the game if his number is greater than Frosty's. Compute the probability that Spot wins.

Problem 2. Ben is throwing darts at a circular target with diameter 10. Ben never misses the target when he throws a dart, but he is equally likely to hit any point on the target. Ben gets $\lceil 5 - x \rceil$ points for having the dart land x units away from the center of the target. What is the expected number of points that Ben can earn from throwing a single dart?

Problem 3. Eight people are posing together in a straight line for a photo. Alice and Bob must stand next to each other, and Claire and Derek must stand next to each other. How many different ways can the eight people pose for their photo?

Problem 4. How many squares are formed by the boundary lines of a regular (8×8) chessboard?

Problem 5. There are 10 Chens in a room, called Chen 1, Chen 2, $\ldots$, Chen 10. An (unordered) subset of Chens is *competent* if there is a Chen in the subset whose number is equal to the number of members of the subset. For example, the subset consisting of Chen 3, Chen 7, and Chen 10 is competent because there are 3 members in the subset and Chen 3 is among them. A competent subset is a *bastion of competence* if no proper subsets of it are competent. Find the total number of bastions of competence.

Problem 6. A 3×6 grid is to be populated with the numbers in the list $\{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9\}$ according to the following rules: (1) Both the first three columns and the last three columns contain the integers 1 through 9. (2) Each number in a given row of the grid is unique. Let N be the number of ways to populate the grid and let k be the largest positive integer such that 2^k divides N. What is k?

Problem 7. Given the digits 1 through 7, one can form $7! = 5040$ numbers by forming different permutations of the 7 digits (for example, 1234567 and 6321475 are two such permutations). If the 5040 numbers obtained are then placed in ascending order what is the 2013^{th} number?

Problem 8. A binary sequence is a sequence of 0's and 1's. We say that A is a subsequence of B if we can make A by deleting elements of B. Let A be the binary sequence 11111011101. How many binary sequences of length 14 have A as a subsequence?

Problem 9. Compute the number of positive integers b where $b \leq 2013$ such that there exists some positive integer N such that $\dfrac{N}{17}$ is a perfect 17th power, $\dfrac{N}{18}$ is a perfect 18th power, and $\dfrac{N}{b}$ is a perfect bth power.

Problem 10. In the National Factoring League (NFL), teams compete head-to-head to factor numbers quickly. There are n teams, and each team plays each other team exactly once. The outcome of each game is a victory for one team and a loss for the other. Assume that all teams are equally good, so they win or lose each game with probability $1/2$. Find the probability that after all games are completed, no two teams have the same record (number of wins and losses).