1. A standard 12-hour clock has hour, minute, and second hands. How many times do two hands cross between 1:00 and 2:00 (not including 1:00 and 2:00 themselves)?

   Answer: 119

   Solution: We know that the hour and minute hands cross exactly once. Let m be the number of minutes past one o'clock that this happens. The angle between the minute hand and the 12 must be equal to the angle between the hour hand and the 12. Since 1 minute is $\frac{360°}{60} = 6°$ on the clock and 1 hour is $\frac{360°}{12} = 30°$, we have $6m = 30(1 + \frac{m}{60})$, so $m = \frac{60}{11} = 5\frac{5}{11}$. Note that the second hand is not at the same position at this time, so we do not have to worry about a triple crossing.

   On the other hand, the second hand crosses the hour hand once every minute, for a total of 60 crossings. Also, the second hand crosses the minute hand once every minute except the first and last, since those crossings take place at 1:00 and 2:00, for a total of 58 crossings. There is a grand total of $1 + 60 + 58 = \boxed{119}$ crossings.

2. Define a set of positive integers to be *balanced* if the set is not empty and the number of even integers in the set is equal to the number of odd integers in the set. How many strict subsets of the set of the first 10 positive integers are balanced?

   Answer: 250

   Solution: The set of the first ten positive integers contains five odd integers and five integers. Therefore, there are $\binom{5}{k}$ ways to choose k odd integers from five odd integers, and also there are $\binom{5}{k}$ ways to choose k even integers from five even integers. Therefore, there are $\binom{5}{k}^2$ ways to pick a balanced subset containing k odd integers and k even integers.

   Therefore, the answer is $\displaystyle\sum_{i=1}^{4} \binom{5}{i}^2 = 5^2 + 10^2 + 10^2 + 5^2 = \boxed{250}$.

3. How many ordered sequences of 1's and 3's sum to 16? (Examples of such sequences are $\{1, 3, 3, 3, 3, 3\}$ and $\{1, 3, 1, 3, 1, 3, 1, 3\}$.)

   Answer: 277

   Solution: Notice that there are 6 sets of 1's and 3's that sum to 16. For a given set suppose there are n 3's we have a total of $(16 - 3n) + n = 16 - 2n$ numbers so we want to compute $\binom{16-2n}{n}$. Hence the total number of possible sequences is:

$$\sum_{n=0}^{5} \binom{16-2n}{n} = \boxed{277}.$$

4. How many positive numbers up to and including 2012 have no repeating digits?

   Answer: 1242

   Solution: All one-digit numbers have no repeating digits, so that gives us 9 numbers. For a two-digit number to have no repeating digits, the first digit must be between 1 and 9, while the second digit must not be equal to the first, giving us $9 \cdot 9 = 81$ numbers. For a three-digit number to have no repeating digits, the first digit must be between 1 and

9, the second digit must not be equal to the first, and the third digit must not be equal to either of the two, giving us $9 \cdot 9 \cdot 8 = 648$ numbers. For a four-digit number between 1000 and 1999 to have no repeating digits, the first digit must be 1, the second digit must not be equal to the first, and so on, giving us $1 \cdot 9 \cdot 8 \cdot 7 = 504$ numbers. Finally, there are no numbers between 2000 and 2012 inclusive with no repeating digits, so the total is $9 + 81 + 648 + 504 = \boxed{1242}$.

5. Define a number to be *boring* if all the digits of the number are the same. How many positive integers less than 10000 are both prime and boring?

   Answer: 5

   Solution: The one-digit boring primes are 2, 3, 5, and 7. The only two-digit boring prime is 11, since 11 divides all other two-digit boring numbers. No three-digit boring numbers are prime, since 111 divides all of them and $111 = 3 \times 37$. No four-digit boring numbers are prime since they are all divisible by 11. Therefore, there are $\boxed{5}$ positive integers less than 10000 which are both prime and boring.

6. A permutation of the first n positive integers is quadratic if, for some positive integers a and b such that $a + b = n$, $a \neq 1$, and $b \neq 1$, the first a integers of the permutation form an increasing sequence and the last b integers of the permutation form a decreasing sequence, or if the first a integers of the permutation form a decreasing sequence and the last b integers of the permutation form an increasing sequence. How many permutations of the first 10 positive integers are quadratic?

   Answer: 1020

   Solution: Clearly, either 1 or 10 must be in the middle of the permutation. Assume without loss of generality that 10 is; we can construct an equivalent permutation with 1 in the middle by replacing each number i with $11 - i$. We can pick any nonempty strict subset of the first 9 positive integers, sort it, place it at the beginning of the permutation, then place 10, then place the unchosen numbers in decreasing order. There are $2^9 - 2 = 510$ ways to do this. Therefore, there are $2 \times 510 = \boxed{1020}$ quadratic permutations of the first 10 positive integers.

7. Two different squares are randomly chosen from an 8×8 chessboard. What is the probability that two queens placed on the two squares can attack each other? Recall that queens in chess can attack any square in a straight line vertically, horizontally, or diagonally from their current position.

   Answer: $\frac{13}{36}$

   Solution: All squares that are on the edge of the chessboard can hit 21 squares; there are 28 such squares. Now consider the 6×6 chessboard that is obtained by removing these bordering squares. The squares on the edge of this board can hit 23 squares; there are 20 of these squares. Now we consider the 12 squares on the boundary of the 4×4 chessboard left; each of these squares can hit 25 squares. The remaining 4 can hit 27 squares. The probability then follows as $\frac{21 \times 28 + 23 \times 20 + 25 \times 12 + 27 \times 4}{64 \times 63} = \boxed{\frac{13}{36}}$.

8. A short rectangular table has four legs, each 8 inches long. For each leg Bill picks a random integer x, $0 \le x < 8$ and cuts x inches off the bottom of that leg. After he's cut all four legs, compute the probability that the table won't wobble (i.e. that the ends of the legs are coplanar).

   Answer: $\frac{43}{512}$

   Solution: We can describe a table by a, b, c, d ($1 \le a, b, c, d \le 8$), giving the final lengths of each of the four legs in clockwise order. How much a table is tipped north to south will be determined by the difference between the lengths a, b and c, d, and east to west by the difference between the lengths a, c and b, d. Hence, for the table to not wobble we must have $a - c = b - d \iff a - b = c - d \iff a + d = b + c$.

   We can therefore split into cases based on $S = a + d = b + c$. The number of ordered pairs (x, y) such that $x + y = S$ and $1 \le x, y \le 8$ is $T_S = 8 - |S - 9|$ (similar to adding the values on two 8-sided dice). The number of choices for (a, d) is therefore T_S and the number of choices for (b, c) is T_S, so the number of choices for (a, b, c, d) is T_S^2.

   Summing over all possible values of S this is

$$\begin{aligned} T_2^2 + \dots T_{16}^2 &= (8 - |2 - 9|)^2 + \cdots + (8 - |16 - 9|)^2 \\ &= 1^2 + 2^2 + \cdots + 7^2 + 8^2 + 7^2 + \cdots + 2^2 + 1^2 \\ &= 2\left(1^2 + \cdots + 7^2\right) + 8^2 \\ &= 2 \cdot \frac{7 \cdot 8 \cdot 15}{6} + 8^2 \\ &= 7 \cdot 8 \cdot 5 + 8^2 \\ &= 8\,(7 \cdot 5 + 8)\,. \end{aligned}$$

   Hence, the probability is

$$\frac{8\,(7 \cdot 5 + 8)}{8^4} = \frac{7 \cdot 5 + 8}{8^3} = \boxed{\frac{43}{512}}.$$

9. Two ants are on opposite vertices of a regular octahedron (an 8-sized polyhedron with 6 vertices, each of which is adjacent to 4 others), and make moves simultaneously and continuously until they meet. At every move, each ant randomly chooses one of the four adjacent vertices to move to. Eventually, they will meet either at a vertex (that is, at the completion of a move) or on an edge (that is, in the middle of a move). Find the probability that they meet on an edge.

   Answer: $\frac{2}{11}$

   Solution: If the two ants are not on the same vertex, they can either be on opposite vertices or on adjacent vertices. Let x and y be the probabilities that the ants will eventually meet on an edge when starting out from opposite vertices and from adjacent vertices, respectively. From opposite vertices, one of the ants must move to one of the remaining four vertices, which are all equivalent with respect to the other ant. That ant can either meet the first ant at a vertex, become adjacent to it (two ways to do this), or again become opposite from it. So

$$x = \frac{1}{4}x + \frac{1}{2}y.$$

If the two ants are adjacent, the cases become slightly more complicated. If the first ant moves towards the second ant, the second ant can move towards it (meeting on an edge); otherwise they will be adjacent. If the first ant moves away from the second ant, they will become adjacent no matter what the second ant does. If the first ant moves to the side (two ways to do this), they will be opposite if the second ant chooses the other direction, and will meet at a vertex if it chooses the same direction. Otherwise they will be adjacent. So

$$y = \frac{1}{8}x + \frac{11}{16}y + \frac{1}{16}.$$

This system of equations is easily solved to obtain $x = \boxed{\frac{2}{11}}$.

10. We say that two polynomials $F(x)$ and $G(x)$ are equivalent mod 5 if and only if $F(x) - G(x) = 5 \cdot H(x)$ for some integer polynomial $H(x)$. We say that $F(x)$ has n as a root mod 5 if and only if $5 \mid F(n)$. How many inequivalent integer polynomials mod 5 of degree at most 3 do not have any integer roots mod 5?

Answer: 204

Solution: Observe that a polynomial

$$I_a(X) = 1 - (X - a)^{p-1}$$

takes value 1 at a and 0 elsewhere in mod p, by Fermat's little theorem. Thus for any polynomial F mod p, we have

$$F(n) = \sum_{a=0}^{p-1} F(a) I_a(n) \pmod{p}$$

for all n. Now the polynomial of degree $\leq p-1$

$$F(X) - \sum_{a=0}^{p-1} F(a) I_a(X)$$

has $0, 1, \cdots, (p-1)$ as roots, thus it should be zero mod p. This means that polynomials mod p of degree less than p have one-to-one correspondence to p-tuples of $(F(0), F(1), \cdots, F(p-1))$ mod p. Since F not having any roots is equivalent to that none of $F(a)$ is zero, there are $(p-1)^p$ ways to choose $(F(0), F(1), \cdots, F(p-1))$. This gives the answer to the first part.

For the second part, note that coefficient of X^{p-1} in $\sum_{a=0}^{p-1} F(n)I_a(X)$ is $-\sum F(a)$, so it is equivalent to find number of p-tuples $(F(0), F(1), \cdots, F(p-1))$ satisfying $F(a) \neq 0 \pmod{p}$ for all a and $\sum F(a) = 0 \pmod{p}$. We define

$$\begin{aligned} A_n = &\text{ the number of } n \text{ tuples } (a_1, \cdots, a_n) \text{ satisfying} \\ &1 \leq a_i \leq p-1, \quad p \mid a_1 + \cdots + a_n \end{aligned}$$

and the problem is to find A_p. We establish the recurrence relation on A_n. For the initial condition we have $A_1 = 0$ and $A_2 = p - 1$. For $n > 2$, note that if $(a_1, a_2, \cdots, a_n)$ is counted in A_n, then a_n is uniquely chosen to be $\equiv -(a_1 + \cdots + a_{n-1}) \pmod{p}$ unless $a_1 + \cdots + a_{n-1}$ is not divisible by p. This is equivalent to say that A_n is same as the number of $(n-1)$-tuples with their sum not divisible by p. This gives the recurrence

$$A_n = (p-1)^{n-1} - A_{n-1}$$

and by solving it we have

$$A_n = (p-1)^{n-1} - (p-1)^{n-2} + (p-1)^{n-3} - \cdots + (-1)^{n-2}(p-1).$$

So the answer is $A_p = \frac{(p-1)^{np} + (-1)^{p-1}(p-1)}{(p-1)+1} = \frac{(p-1)^p - (p-1)}{p}$. Evaluating at $p = 5$, we get $\frac{4^5 - 4}{5} = \frac{1020}{5} = \boxed{204}$.