

1. ABC is an equilateral triangle with side length 1. Point D lies on \overline{AB} , point E lies on \overline{AC} , and points G and F lie on \overline{BC} , such that $DEFG$ is a square. What is the area of $DEFG$?

Answer: $21 - 12\sqrt{3}$

Solution: Let x be the length of a side of square $DEFG$. Then $DE = EF = x$. Note that $\triangle ADE$ is equilateral since $\overline{DE} \parallel \overline{BC}$ and hence $\triangle ADE \sim \triangle ABC$, so $AE = DE = x$, and consequently $EC = 1 - x$. Since $\triangle ECF$ is a $30^\circ - 60^\circ - 90^\circ$ triangle, we have the proportion

$$\frac{EF}{EC} = \frac{x}{1-x} = \frac{\sqrt{3}}{2},$$

so $x = \frac{\sqrt{3}}{2+\sqrt{3}} = 2\sqrt{3} - 3$. Hence the area of $DEFG$ is $x^2 = \boxed{21 - 12\sqrt{3}}$.

2. A circle with radius 1 has diameter AB . C lies on this circle such that $\widehat{AC} / \widehat{BC} = 4$. \overline{AC} divides the circle into two parts, and we will label the smaller part Region I. Similarly, \overline{BC} also divides the circle into two parts, and we will denote the smaller one as Region II. Find the positive difference between the areas of Regions I and II.

Answer: $\frac{3\pi}{10}$

Solution: Let O be the center of the circle. Note that CO bisects AB , so the areas of $\triangle ACO$ and $\triangle BCO$ are equal. Hence, the desired difference in segment areas is equal to the difference in the areas of the corresponding sectors. The sector corresponding to \widehat{AC} has area $\frac{2\pi}{5}$, and the sector corresponding to \widehat{BC} has area $\frac{\pi}{10}$, so the desired difference is

$$\boxed{\frac{3\pi}{10}}.$$

3. In trapezoid $ABCD$, $BC \parallel AD$, $AB = 13$, $BC = 15$, $CD = 14$, and $DA = 30$. Find the area of $ABCD$.

Answer: 252

Solution: We can use the standard method of setting up a two-variable system and solving for the height of the trapezoid. However, since one base is half the length of the other, we may take a shortcut. Extend AB and CD until they meet at E . Clearly, BC is a midline of triangle EAD , so we have $EA = 2BA = 26$ and $ED = 2CD = 28$. The area of EAD is therefore four times that of a standard 13-14-15 triangle, which we know is $\frac{1}{2} \cdot 14 \cdot 12 = 84$ (since the altitude to the side of length 14 splits the triangle into 9-12-15 and 5-12-13 right triangles). The area of the trapezoid is $\frac{3}{4}$ the area of EAD by similar triangles, and is therefore $3 \cdot 84 = \boxed{252}$.

A similar solution draws lines from B and C to the midpoint of AD to form three 13-14-15 triangles.

4. Circle O has radius 18. From diameter AB , there exists a point C such that BC is tangent to O and AC intersects O at a point D , with $AD = 24$. What is the length of BC ?

Answer: $18\sqrt{5}$

Solution: Since $\angle ADB = \angle ABC = 90^\circ$, $\triangle ABC \sim \triangle ADB$. In particular, $\frac{AB}{AD} = \frac{AC}{AB}$, so $AC = \frac{AB^2}{AD}$. Therefore, $AC = \frac{36^2}{24} = 54$. Since $AD = 24$, $DC = 30$. By Power of a Point, $BC = \sqrt{30 \times 54} = \boxed{18\sqrt{5}}$.

5. Let ABC be an equilateral triangle with side length 1. Draw three circles O_a , O_b , and O_c with diameters BC , CA , and AB , respectively. Let S_a denote the area of the region inside O_a and outside of O_b and O_c . Define S_b and S_c similarly, and let S be the area of the region inside all three circles. Find $S_a + S_b + S_c - S$.

Answer: $\frac{\sqrt{3}}{2}$

Solution: Let x be $1/4$ the area of ABC , and let y be the area of a 60 degree sector of O_a minus x . Note that

$$S_a = S_b = S_c = 3x + y, \quad S = x + 3y,$$

$$\text{so } S_a + S_b + S_c - S = 8x = 2|\triangle ABC| = \boxed{\sqrt{3}/2}.$$

6. Let $ABCD$ be a rectangle with area 2012. There exist points E on AB and F on CD such that $DE = EF = FB$. Diagonal AC intersects DE at X and EF at Y . Compute the area of triangle EXY .

Answer: $\frac{503}{6}$

Solution: Let (XYZ) denote the area of triangle XYZ .

After a bit of angle-chasing, we can use SAS congruence to prove that $\triangle DEF \cong \triangle BFE$, so $EB \cong DF$ and therefore $AE \cong FC$. If we draw altitudes from E and F onto CD and AB , respectively, we note that $2AE = 2FC = DF = BE$, so $AE = \frac{1}{3}AB$.

Next, note that $\triangle AEX \sim \triangle CDX$, so $\frac{CX}{AX} = \frac{CD}{AE} = 3$. Also, $\triangle CFY \sim \triangle AEY$, so $\frac{CY}{AY} = \frac{CF}{AE} = 1$. Hence, $XY = \frac{1}{4}AC \implies (EXY) = \frac{1}{4}(EAC)$.

Finally, $AE = \frac{1}{3}AB \implies (EAC) = \frac{1}{3}(BAC)$. Since BAC is half the rectangle and therefore has area 1006, we get $(EXY) = \frac{1006}{12} = \boxed{\frac{503}{6}}$.

7. What is the radius of the largest sphere that fits inside an octahedron of side length 1?

Answer: $\frac{1}{\sqrt{6}}$

Solution: It is obvious that the sphere must be tangent to each face, because if not, then it can be moved so that it is tangent to four faces; now the radius can be increased until the sphere is tangent to the other four. Additionally, it is clear that the center of the sphere should be in the center of the octahedron.

Now notice that the sphere must be tangent to the octahedron at the centroid of each face. This can be seen by symmetry. It is clear that it should be tangent somewhere along the median from one vertex to the opposite side, and this is true for all three medians, which meet at the centroid.

Now we can proceed in a few ways. One way is to isolate one half of the octahedron i.e. a square-based pyramid. Slice this pyramid in half perpendicular to the square base and parallel to one of the sides of the square base. This slice will go through the medians of two opposite triangular faces, in addition to the center of the sphere itself. Hence, we get an isosceles triangle ABC with base $BC = 1$ and legs of length $\sqrt{3}/2$. O , the center of the sphere, is the midpoint of BC . The radius of the sphere is the altitude from O to AB . If this altitude intersects AB at D , then we have

$$OD \cdot AB = AO \cdot BO,$$

since both equal twice the area of AOB , and so $DO = \frac{AO \cdot BO}{AB} = \frac{(1/\sqrt{2})(1/2)}{\sqrt{3}/2} = \boxed{1/\sqrt{6}}$.

Alternatively, note that our octahedron can be obtained by reflecting the region $x+y+z \leq 1/\sqrt{2}$, $x, y, z \geq 0$ across the xy , yz , and zx planes. The inscribing sphere has its center at origin, so its radius is the distance from the origin to the plane $x+y+z = 1/\sqrt{2}$, which is $1/\sqrt{6}$.

8. A red unit cube $ABCDEFGH$ (with E below A , F below B , etc.) is pushed into the corner of a room with vertex E not visible, so that faces $ABFE$ and $ADHE$ are adjacent to the wall and face $EFGH$ is adjacent to the floor. A string of length 2 is dipped in black paint, and one of its endpoints is attached to vertex A . How much surface area on the three visible faces of the cube can be painted black by sweeping the string over it?

Answer: $\frac{2\pi}{3} + \sqrt{3} - 1$

Solution: First, it is clear that all of face $ABCD$ can be painted black. This has area 1.

Now we look at the other two visible faces. By symmetry, we only need to consider one of these faces, say $BCGF$. Unfold $BCGF$ along BC so that it is coplanar with $ABCD$, forming a rectangle $AF'G'D$ with width 1 and height 2. Now, it is clear that the region that can be painted on $BCGF$ is precisely the part of $BCC'F'$ that is at most two units away from A . Let a circle centered at A with radius two intersect DG' at X . Since $AX = 2$, $AD = 1$, and $AD \perp XD$, we conclude that $m\angle DAX = \frac{\pi}{3} \implies \angle F'AX = \frac{\pi}{6}$. Letting $(P_1P_2 \dots P_n)$ denote the area of the n -gon with vertices P_1, \dots, P_n , we can write the desired area as

$$\text{area of sector } F'AX + (AXD) - (ABCD) = \frac{2^2\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

Putting all this together, we get our final answer to be

$$1 + 2 \left(\pi/3 + \frac{\sqrt{3}}{2} - 1 \right) = \boxed{\frac{2\pi}{3} + \sqrt{3} - 1}.$$

9. Let ABC be a triangle with incircle O and side lengths 5, 8, and 9. Consider the other tangent line to O parallel to BC , which intersects AB at B_a and AC at C_a . Let r_a be the inradius of triangle AB_aC_a , and define r_b and r_c similarly. Find $r_a + r_b + r_c$.

Answer: $\frac{6\sqrt{11}}{11}$

Solution: We claim that the answer is equal to the inradius in general. Let $T_a = AB_aC_a$, $T_b = A_bBC_b$, $T_c = A_cB_cC$ be the smaller triangles cut by the tangents drawn to O . Also let D , E , and F be the points of tangency between O and BC , CA , and AB respectively. By considering the fact that tangents to O from the same point should have the same length, we have $AB_a + B_aC_a + C_aA = AE + AF$. If we sum this over all vertices, then we can see that the sum of the perimeters of T_a , T_b , and T_c equals the perimeter of A . Then, the Principle of Similarity gives $r_a + r_b + r_c = r$ where r is inradius of ABC . The inradius can be calculated by Heron's Formula as

$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \frac{\sqrt{11 \cdot 6 \cdot 3 \cdot 2}}{11} = \boxed{\frac{6\sqrt{11}}{11}}.$$

Alternatively, let h_a denote the height of the altitude from A to BC , and let r be the inradius of ABC . Since $\triangle ABC \sim \triangle AB_aC_a$ and since the altitude from A to B_aC_a has length $h_a - 2r$, we get

$$\frac{r_a}{r} = \frac{h_a - 2r}{h_a}.$$

Noticing that

$$r = \frac{(ABC)}{\frac{1}{2}(a+b+c)} = \frac{ah_a}{a+b+c},$$

we get

$$r_a = r - 2r \frac{a}{a+b+c}.$$

Applying the same reasoning to r_a and r_b , we can compute

$$r_a + r_b + r_c = 3r - 2r = r.$$

10. A large flat plate of glass is suspended $\sqrt{2/3}$ units above a large flat plate of wood. (The glass is infinitely thin and causes no funny refractive effects.) A point source of light is suspended $\sqrt{6}$ units above the glass plate. An object rests on the glass plate of the following description. Its base is an isosceles trapezoid $ABCD$ with $AB \parallel DC$, $AB = AD = BC = 1$, and $DC = 2$. The point source of light is directly above the midpoint of CD . The object's upper face is a triangle EFG with $EF = 2$, $EG = FG = \sqrt{3}$. G and AB lie on opposite sides of the rectangle $EFCD$. The other sides of the object are $EA = ED = 1$, $FB = FC = 1$, and $GD = GC = 2$. Compute the area of the shadow that the object casts on the wood plate.

Answer: $4\sqrt{3}$

Solution: We have $\angle A = \angle B = 120^\circ$ and $\angle C = \angle D = 60^\circ$ at the base, and the three "side" faces – ADE , BCF , and CDG – are all equilateral triangles. If those faces are folded down to the glass plate, they will form a large equilateral triangle of side length 3. Let E_0 , F_0 , and G_0 be the vertices of this equilateral triangle corresponding to E , F and G , respectively; the large triangle can be folded up along AD , CD , and BD respectively to form the three side faces of the object.

Observe that M , the midpoint of CD , is the centroid of $E_0F_0G_0$. As side ADE is folded along AD , which is perpendicular to E_0M , the projection of E onto the glass plate still lies on EM . This also holds for the projections of F and G , so projections E_1 , F_1 , and G_1 of E , F , and G lie on E_0M , F_0M and G_0M respectively.

Since $EFCD$ is a rectangle, E_1F_1CD is as well. Thus E_1D is perpendicular to EA . From E_0E_1 being perpendicular to AD we can conclude that E_1 should be the center of triangle ADE_0 . Symmetry gives $AE = DE = E_0E$, so AE_0DE should be a regular tetrahedron. A similar argument applies to BF_0CF .

The next step is to figure out the location of G . As $EG = \sqrt{3}$ and $DG = \sqrt{2}$, it follows that $\angle DEG$ is right. Similarly $\angle CFG$ is also right, so plane EFG should be perpendicular to plane $EFCD$.

Now we cut the whole object along the perpendicular bisector plane of AB and consider its cross-section along the plane. It will cut AB and EF along their midpoints N and P respectively. As $ABMP$ forms a regular tetrahedron of side length 1 and N is midpoint of AB , we have $NM = NP = \sqrt{3}/2$. Also $MG = \sqrt{3}$ and $\angle MPG$ is right. Let Q be the midpoint of MG ; then $PQ = MQ = \sqrt{3}/2$, since right triangles are inscribed in semicircles. It follows that NPM and QMP are congruent and NP and MG are parallel. From $MG = MG_0 = \sqrt{3}$ and $NP = NM = \sqrt{3}/2$, this gives similarity between NMP and MG_0G , and $GG_0 = 2PM = 2$. Therefore $DCGG_0$ also forms a regular tetrahedron.

Since AE_0DE , BF_0CF , and CG_0DG are all regular tetrahedrons, we have three lines E_0E , F_0F , and G_0G meeting at a point X where $E_0F_0G_0X$ forms a regular tetrahedron of side length 3. Thus we finally demystified our object completely: it was obtained by cutting the regular tetrahedron $E_0F_0G_0X$ along planes EFG , ADE , BCF , CDG . Moreover we find that X is actually our point source, as it is also directly above M - both the midpoint of CD and the center of $E_0F_0G_0$ - and its height is $\sqrt{6}$, the same as that of point source. So the projection of the object to the glass plate will be exactly $E_0F_0G_0$, an equilateral triangle of side length 3. Hence the projection down to the wood plate will give an equilateral triangle of side length 4, and our answer is its area, $\boxed{4\sqrt{3}}$.