1. Define a set of positive integers to be *balanced* if the set is not empty and the number of even integers in the set is equal to the number of odd integers in the set. How many strict subsets of the set of the first 10 positive integers are balanced?

   Answer: 250

   Solution: The set of the first ten positive integers contains five odd integers and five integers. Therefore, there are $\binom{5}{k}$ ways to choose k odd integers from five odd integers, and also there are $\binom{5}{k}$ ways to choose k even integers from five even integers. Therefore, there are $\binom{5}{k}^2$ ways to pick a balanced subset containing k odd integers and k even integers. Therefore, the answer is $\sum_{i=1}^{4} \binom{5}{i}^2 = 5^2 + 10^2 + 10^2 + 5^2 = \boxed{250}$.

2. At the 2012 Silly Math Tournament, hamburgers and hot dogs are served. Each hamburger costs \$4 and each hot dog costs \$3. Each team has between 6 and 10 members, inclusive, and each member buys exactly one food item. How many different values are possible for a team's total food cost?

   Answer: 23

   Solution: The minimum food cost for a team is $6(\$3) = \$18$, and the maximum food cost is $10(\$4) = \$40$. Note that all intermediate values can be achieved. Suppose n dollars can be achieved by purchasing a hamburgers and b hot dogs, where $18 \leq n < 40$. If $b > 0$, then $n + 1$ dollars can be achieved by purchasing $a + 1$ hamburgers and $b - 1$ hot dogs. If $b = 0$, then $n + 1$ dollars can be achieved by purchasing $a - 2$ hamburgers and $b + 3$ hot dogs. (This increases the number of team members by 1.) Repeating this process until \$40 is reached, the number of team members cannot decrease, and since we end up with 10 team members, the number of team members is always contained within 6 and 10.

   Hence the number of different values is $40 - 18 + 1 = \boxed{23}$.

3. How many ordered sequences of 1's and 3's sum to 16? (Examples of such sequences are $\{1, 3, 3, 3, 3, 3\}$ and $\{1, 3, 1, 3, 1, 3, 1, 3\}$.)

   Answer: 277

   Solution: Notice that there are 6 sets of 1's and 3's that sum to 16. For a given set suppose there are n 3's we have a total of $(16 - 3n) + n = 16 - 2n$ numbers so we want to compute $\binom{16-2n}{n}$. Hence the total number of possible sequences is:

$$\sum_{n=0}^{5} \binom{16-2n}{n} = \boxed{277}.$$

4. How many positive numbers up to and including 2012 have no repeating digits?

   Answer: 1242

   Solution: All one-digit numbers have no repeating digits, so that gives us 9 numbers. For a two-digit number to have no repeating digits, the first digit must be between 1 and 9, while the second digit must not be equal to the first, giving us $9 \cdot 9 = 81$ numbers. For a three-digit number to have no repeating digits, the first digit must be between 1 and 9, the second digit must not be equal to the first, and the third digit must not be equal to either of the two, giving us $9 \cdot 9 \cdot 8 = 648$ numbers. For a four-digit number between 1000 and 1999 to have no repeating digits, the first digit must be 1, the second digit must not be equal to the first, and so on, giving us $1 \cdot 9 \cdot 8 \cdot 7 = 504$ numbers. Finally, there are no numbers between 2000 and 2012 inclusive with no repeating digits, so the total is $9 + 81 + 648 + 504 = \boxed{1242}$.

5. ABC is an equilateral triangle with side length 1. Point D lies on $\overline{AB}$, point E lies on $\overline{AC}$, and points G and F lie on $\overline{BC}$, such that $DEFG$ is a square. What is the area of $DEFG$?

   Answer: $\mathbf{21 - 12\sqrt{3}}$

   Solution: Let x be the length of a side of square $DEFG$. Then $DE = EF = x$. Note that $\triangle ADE$ is equilateral since $\overline{DE} \parallel \overline{BC}$ and hence $\triangle ADE \sim \triangle ABC$, so $AE = DE = x$, and consequently $EC = 1 - x$. Since $\triangle ECF$ is a $30° - 60° - 90°$ triangle, we have the proportion
$$\frac{EF}{EC} = \frac{x}{1-x} = \frac{\sqrt{3}}{2},$$
   so $x = \frac{\sqrt{3}}{2+\sqrt{3}} = 2\sqrt{3} - 3$. Hence the area of $DEFG$ is $x^2 = \boxed{21 - 12\sqrt{3}}$.

6. If f is a monic cubic polynomial with $f(0) = -64$, and all roots of f are non-negative real numbers, what is the largest possible value of $f(-1)$? (A polynomial is monic if it has a leading coefficient of 1.)

   Answer: $\mathbf{-125}$

   Solution: If the three roots of f are r_1, r_2, r_3, we have $f(x) = x^3 - (r_1+r_2+r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3$, so $f(-1) = -1 - (r_1 + r_2 + r_3) - (r_1r_2 + r_1r_3 + r_2r_3) - r_1r_2r_3$. Since $r_1r_2r_3 = 64$, the arithmetic mean-geometric mean inequality reveals that $r_1 + r_2 + r_3 \geq 3(r_1r_2r_3)^{1/3} = 12$ and $r_1r_2 + r_1r_3 + r_2r_3 \geq 3(r_1r_2r_3)^{2/3} = 48$. It follows that $f(-1)$ is at most $-1 - 12 - 48 - 64 = \boxed{-125}$. We have equality when all roots are equal, i.e. $f(x) = (x-4)^3$.

7. In trapezoid $ABCD$, $BC \parallel AD$, $AB = 13$, $BC = 15$, $CD = 14$, and $DA = 30$. Find the area of $ABCD$.

   Answer: 252

   Solution: We can use the standard method of setting up a two-variable system and solving for the height of the trapezoid. However, since one base is half the length of the other, we may take a shortcut. Extend AB and CD until they meet at E. Clearly, BC is a midline of triangle EAD, so we have $EA = 2BA = 26$ and $ED = 2CD = 28$. The area of EAD is therefore four times that of a standard 13-14-15 triangle, which we know is $\frac{1}{2} \cdot 14 \cdot 12 = 84$ (since the altitude to the side of length 14 splits the triangle into 9-12-15 and 5-12-13 right triangles). The area of the trapezoid is $\frac{3}{4}$ the area of EAD by similar triangles, and is therefore $3 \cdot 84 = \boxed{252}$.

A similar solution draws lines from B and C to the midpoint of AD to form three $13-14-15$ triangles.

8. Circle O has radius 18. From diameter AB, there exists a point C such that BC is tangent to O and AC intersects O at a point D, with $AD = 24$. What is the length of BC?

   Answer: $\mathbf{18\sqrt{5}}$

   Solution: Since $\angle ADB = \angle ABC = 90°$, $\triangle ABC \sim \triangle ADB$. In particular, $\dfrac{AB}{AD} = \dfrac{AC}{AB}$, so $AC = \dfrac{AB^2}{AD}$. Therefore, $AC = \dfrac{36^2}{24} = 54$. Since $AD = 24$, $DC = 30$. By Power of a Point, $BC = \sqrt{30 \times 54} = \boxed{18\sqrt{5}}$.

9. The quartic (4th-degree) polynomial $P(x)$ satisfies $P(1) = 0$ and attains its maximum value of 3 at both $x = 2$ and $x = 3$. Compute $P(5)$.

   Answer: $\mathbf{-24}$

   Solution: Consider the polynomial $Q(x) = P(x) - 3$. Q has roots at $x = 2$ and $x = 3$. Moreover, since these roots are maxima, they both have multiplicity 2. Hence, Q is of the form $a(x-2)^2(x-3)^2$, and so $P(x) = a(x-2)^2(x-3)^2 + 3$. $P(1) = 0 \implies a = -\frac{3}{4}$, allowing us to compute $P(5) = -\frac{3}{4}(9)(4) + 3 = \boxed{-24}$.

10. Compute the ordered pair of real numbers (a, b) such that $a < k < b$ if and only if $x^3 + \dfrac{1}{x^3} = k$ does not have a real solution in x.

    Answer: $\mathbf{(-2, 2)}$

    Solution: Substitute $y = x^3$, so now we want to find the values of k such that $y + \dfrac{1}{y} = k$ has no real solutions in y. In particular, since $y = x^3$ is an invertible function, $x^3 + \dfrac{1}{x^3} = k$ does not have a real solution in x if and only if $y + \dfrac{1}{y} = k$ has no real solutions in y. Therefore, clearing denominators of $y + \dfrac{1}{y} = k$ gives us $y^2 + 1 = ky$, so $y^2 - ky + 1 = 0$, so this quadratic equation has no solutions when the discriminant is negative. The discriminant is $k^2 - 4$, which is negative when $-2 < k < 2$, so therefore the ordered pair is $\boxed{(-2, 2)}$.