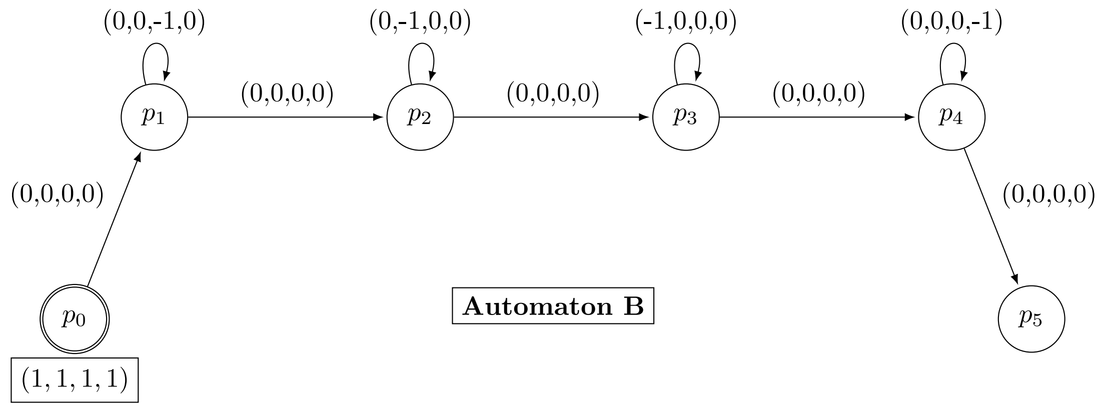
13^{th} Annual Johns Hopkins Math Tournament
Saturday, February 19, 2011

Automata Theory EUR solutions

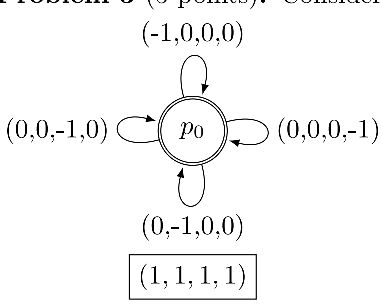
Problem 1 (5 points)**.** Prove that any surjective map between finite sets of the same cardinality is a bijection.
SOLUTION:
Let X and Y be two finite sets of equal cardinality. Then for all $y \in Y$, there exists an $x \in X$ such that $f(x) = y$ by definition of surjection.
What this is saying is that every y in Y has an associated x in X.
But every x also needs to have an associated y in Y, because both sets have finite and equal cardinality. Suppose otherwise for the sake of contradiction. Then X must have cardinality greater than Y, since it will have all of the elements associated with those in Y plus at least one element, x, that is not mapped to Y. Thus, the cardinalities of both sets would have to be different, a contradiction.
Hence the function is injective, and thus a bijection.

Problem 2 (5 points)**.** We can now return to automaton B. Is B isomorphic to A? Prove your assertion.



SOLUTION:
Yes.
$q_0 \to p_0$
$q_1 \to p_3$
$q_2 \to p_2$
$q_3 \to p_1$
$q_4 \to p_4$
$q_5 \to p_5$
All transitions are $(0, 0, 0, 0)$ in both cases, there are the same number of transitions in both cases, and each vertex has only one transition.

Problem 3 (5 points)**.** Consider automaton C, represented below. Prove that it is *not* isomorphic to A.

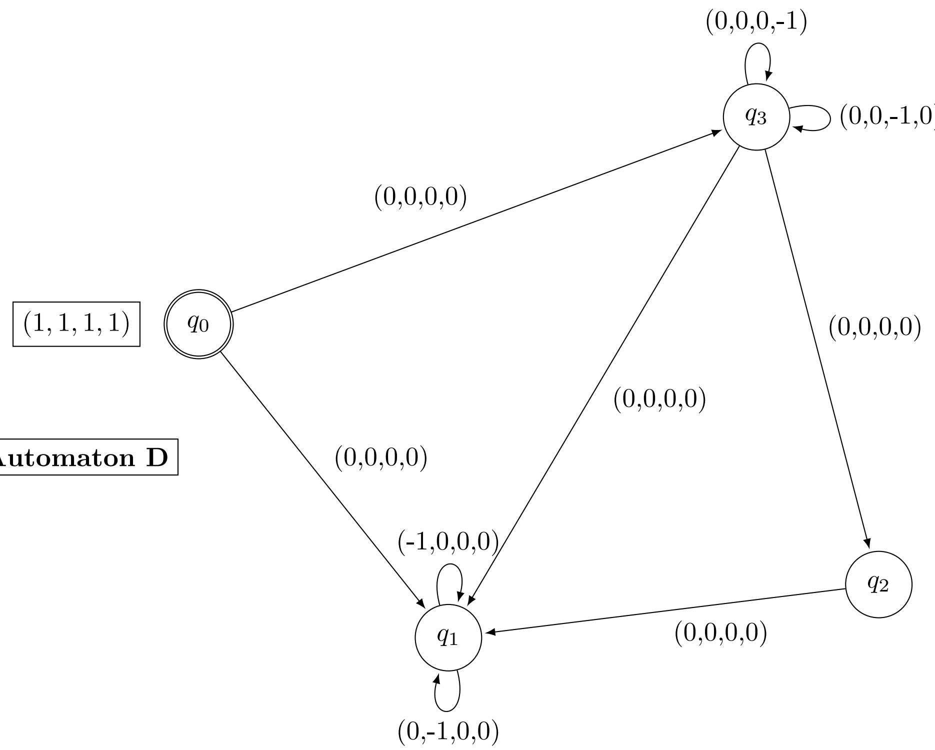


Automaton C

SOLUTION: Short answer: they are not isomorphic because they do not have the same number of transitions.

Longer answer: Give both automaton one move. A can only reach $(1,1,1,1)$ by not moving, or $(0,1,1,1)$ by going to q_1. C can reach more states, for example $(1,0,1,1)$.

Problem 4 (5 points)**.** apply rules 1 and 2 to Automaton A to simplify the drawing. (Sketch it)
SOLUTION:
Exactly the same as automaton C.



Problem 5 (5 points)**.**
SOLUTION:
Yes, E is equivalent to A. You can draw the specific stages, and order is not important here, but for example one can merge q_0 with q_1, then q_1 with q_3, then q_3 with q_2, and you end up with four loops on a single vertex q_2 labelled $(-1,0,0,0)$, $(0,-1,0,0)$, $(0,0,-1,0)$, and $(0,0,0,-1)$, with the same starting 4-tuple $(1,1,1,1)$.

Problem 6 (6 points, 3 for each direction)**.** Consider the set of all VASS automata equivalent to A. Call this set Ξ. Now consider the set of all isomorphic VASS automata to A. Call this set Υ.
Is Ξ a subset of Υ? Is Υ a subset of Ξ? (Prove or disprove)
SOLUTION:
By problem 2 and 3, Ξ (set of equivalent automata to A) is not a subset of Υ (set of isomorphic automata to A) because we have an element of Ξ that is not in Υ; automaton D is not isomorphic to A so is not in Υ, but D is equivalent to A so is in Ξ.
Υ is a subset of Ξ, however, because if an automata is isomorphic to A, call it X, then both have the same loops and transitions, so if any can be removed because they are $(0,0,0,0)$ loops or $(0,0,0,0)$ transitions, they can be removed in both. So both simplified automata are equal, up to isomorphism. So X and A are equivalent.

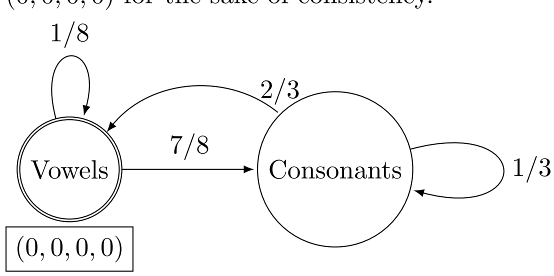
Problem 7 (9 points, 3 for each property)**.** Prove that is-isomorphic-to is an equivalence relation on the set of all VASS automata.
SOLUTION: Let A be a VASS automaton. Apply the identity function that sends every vertex to itself. Then clearly no transitions are changed, and the 'new' automaton is therefore isomorphic to the old. Hence A is isomorphic to A, which means that is-isomorphic-to is reflexive.
Let A and B be two VASS automata. Suppose A is isomorphic to B. Then there is a bijection from the set of vertices of A to the set of vertices of B which preserves transitions. Since this is a bijection, call it

f, there exists an inverse function f^{-1} which maps the set of vertices of B to the set of vertices of A while preserving transitions. So B is isomorphic to A. So is-isomorphic-to is symmetric.
Let A, B, and C be three VASS automata. Suppose A is isomorphic to B, and that B is isomorphic to C. Then there is a bijection f from the set of vertices of A to the set of vertices of B which preserves transitions, and another bijection g from the set of vertices of B to the set of vertices of C which preserves transitions. Now consider the composition of g and f: this is a bijection from the set of vertices of A to the set of vertices of C which preserves transitions. So is-isomorphic-to is transitive.
Since is-isomorphic-to is reflexive, symmetric, and transitive, is-isomorphic-to is an equivalence relation.

Problem 8 (9 points, 3 for each property)**.** Prove that is-equivalent-to is an equivalence relation on the set of all VASS automata.
SOLUTION: Let A be a VASS automaton. Remove any $(0,0,0,0)$ loops and $(0,0,0,0)$ transitions as per the rules for simplifications. Now from problem 6 we know that this simplified automata is isomorphic to itself, so we know A is equivalent to itself. Hence is-equivalent-to is reflexive.
Let A and B be two equivalent automata. Remove any $(0,0,0,0)$ loops and $(0,0,0,0)$ transitions as per the rules for simplifications. Since A and B are equivalent, their simplified versions, call them A' and B' are isomorphic. By problem 6, this means that B' and A' are isomorphic. Which means that B and A are equivalent by definition. So is-equivalent-to is symmetric.
Let A, B, and C be three VASS automata such that A is equivalent to B and B is equivalent to C. Remove any $(0,0,0,0)$ loops and $(0,0,0,0)$ transitions as per the rules for simplifications. Since A is equivalent to B, we know their simplified versions, call them A' and B', are isomorphic. Similarly, we know that B' is isomorphic to C', the simplified version of C. By problem 6, this means that A' is isomorphic to C' by composition of the bijections. So A is equivalent to C. So is-equivalent-to is transitive.
Since is-equivalent-to is reflexive, symmetric, and transitive, is-equivalent-to is an equivalence relation.

Problem 9 (9 points, 3 for each property)**.** If $f : A \to B$ and $g : A \to B$ are injective, then show there exists $h : A \to B$ which is bijective
SOLUTION: First, note that we have not assumed A or B to be finite. Let $X_0 = A - g(B)$. Let $X_{n+1} = g(f(X_n))$. Let $X = \cup_{n=0}^{\infty} X_n$. Define h as follows $h(x) = f(x)$ if $x \in X$, otherwise, $h(x) = g^{-1}(x)$. h is the function with the desired properties. To learn more about this result, as well as to see other proofs, the Wikipedia page for the Cantor-Bernstein-Schroeder theorem is a good source of information.

Problem 10 (6 points, 3 for each part)**.** In 1903, A. A. Markov, reading Pouchkine's poem "Eugene Oneguinde", remarked that the consonants and vowels were succeeding one another according to the following automaton. This particular automaton works exactly like a VASS automaton, with the sole difference that you do not get to choose which vertex to go to; it is determined with the probability indicated on the diagram next to the edges. You will notice that there are no listed transition costs; simply assume that they are all $(0,0,0,0)$ and that therefore you can always make the transitions as required. The initial state is given as $(0,0,0,0)$ for the sake of consistency.



The first letter of the poem is a vowel.

(1) What is the probability that the 2^{nd} letter is a consonant?
(2) What is the probability that the 3^{rd} letter is a consonant?

SOLUTION:

(1) $P(\text{2nd is consonant}|\text{1st is vowel}) = 7/8$
(2) $P(\text{2nd is vowel}|\text{1st is vowel}) = 1/8$, so we have $P(\text{3rd is consonant}|\text{1st is vowel}) = \frac{7}{8}(\frac{1}{3}) + \frac{1}{3}(\frac{1}{8}) = \frac{1}{3}$

Problem 11 (5 points)**.** We consider a particular case of the *reachability problem*: can one ever reach $(4, 4, 4, 4)$, given the following drawing of the VASS automaton in question. The starting 4-tuple is, as indicated, $(4, 3, 2, 1)$.
If yes, give an explicit path to take to reach it in terms of the nodes to visit. (in other words something like ǵo from A to B, to B again, etc.)́ If no, explain why such a configuration can never be reached.
SOLUTION: No, we cannot reach this state with the given VASS automaton. We can never change the second entry of the 4-tuple which is set to 3 and we would need it to be 4.

Problem 12 (5 points)**.** Same question as Problem 8, but with $(1, 4, 0, 4)$.
SOLUTION: Yes, we can reach this state with the given VASS automaton.
$q_0 \to q_1 \to q_2$: net change none, $(1, 4, 0, 4)$.
q_2 loop twice: net change $(1, 4, 4, 4)$.
$q_2 \to q_0$: net change $(3, 4, 4, 4)$.
$q_0 \to q_1$: net change $(4, 4, 4, 4)$.
Stop.

Problem 13 (5 points)**.** Prove that if the tuple (a, b, c, d) is a 4-tuple that is covered by a given automata, then: $a \geq 0$, and $b \geq 0$, and $c \geq 0$, and $d \geq 0$.
SOLUTION:
Suppose otherwise, for the sake of contradiction. Then we have reached a state where at least one value of the 4-tuple is below zero. The starting tuple is strictly positive, so we know this was because of a transition. And this breaks our rule for valid transitions, a contradiction. Hence if the tuple (a, b, c, d) is a 4-tuple that is covered by a given automata, then: $a \geq 0$, and $b \geq 0$, and $c \geq 0$, and $d \geq 0$.

Problem 14 (30 points)**.** Prove the given theorem by Konig (1936): Given a graph G, you can assign a colour to every vertex of G such that no adjacent vertices have the same colour, using only 2 colours, if and only if G has no odd cycles.
SOLUTION: Suppose G has at least one odd cycle. Note that if one cycle in a graph is not 2-colourable, then the entire graph cannot be 2-colourable, since the graph can be decomposed into a finite amount of distinct cycles. Then when we try to 2-colour this odd cycle, we cannot, because an odd number is not divisible by 2, so there will be two adjacent vertices with identical colours. Therefore if G has at least one odd cycle, it is not 2-colourable. Hence, if G is two colourable, then it does not have an odd cycle.
Now suppose G has no odd cycle. Decompose G into its finite amount of distinct even cycles. Now for every even cycle, do the following: assign the first vertex the colour red. Assign the adjacent vertex the colour blue. Repeat until the cycle is complete. Since there are an even number of vertices in the cycle, it is divisible by 2 and we can colour it like we said. Now, the problem begins when we want to re-connect these finite, even cycles. But this is fine, because suppose one of the two connecting vertices between two cycles is black. Then colour the other white. This will not affect their respective even cycles, which can then be coloured like we described. We have now 2-coloured the entire graph.

Problem 15 (20 points)**.** Give an example (their drawings will suffice; use the same notation as for VASS automata, i.e. circles for vertices and lines for edges) of two graphs that have the same number of vertices and the same number of edges but are not isomorphic. Prove your assertion.
SOLUTION:
Consider the cyclic graph on $n(n-1)/2$ vertices, and the graph which consists of the complete graph on n vertices as well as $n - n(n-1)/2$ singleton vertices.

Problem 16 (15 points)**.** What is the embedding in $\mathbb{R}^3$ of the drawing of the complement of the graph whose vertices and edges are those of a cube?
SOLUTION: Give full marks if they give the drawing in $\mathbb{R}^2$, which will be look like two disjoint tetrahedrons. Not necessary for full points, but here are the formal definitions:
We have: $G = (\{a, b, c, d, e, f, g, h\}, \{\{ab\}, \{ad\}, \{af\}, \{bc\}, \{bg\}, \{cd\}, \{ch\}, \{de\}, \{ef\}, \{eh\}, \{fg\}, \{gh\}\})$.
Then the complement is: $\bar{G} = (\{a, b, c, d, e, f, g, h\}, \{\{ac\}, \{ae\}, \{ag\}, \{ah\},$
$\{bd\}, \{be\}, \{bf\}, \{bh\}, \{ce\}, \{cf\}, \{cg\}, \{df\}, \{dg\}, \{dh\}, \{eg\}, \{fh\})$

Problem 17 (20 points)**.** Give an example (their drawings will suffice; use the same notation as for automata, i.e. circles for vertices and lines for edges) of a self-complementary graph with five vertices. (Giving the mapping of vertices of G to vertices in $\bar{G}$ would be nice if you have the time.)
How many edges must a self complimentary graph on n vertices have?
SOLUTION: Consider the cycle on 5 vertices.
If you account for all the edges in the graph and its complement, you get all the edges in the complete graph- this has n(n-1)/2 edges. Since the number of edges in each graph must be the same, the number of edges in the graph must be n(n-1)/4

Problem 18 (25 points)**.** Prove the given Lemma:
$n \equiv 0$ or $n \equiv 1 \mod 4$ Lemma:
For a self-complementary graph with n vertices, either $n \equiv 0 \mod 4$ or $n \equiv 1 mod 4$.
(For those unfamiliar with modulo notation; $n \equiv p \mod 4$ means the remainder of n divided by p is 0)
SOLUTION: Consider a graph C with n vertices. The total number of possible edges is $(n-1)+(n-2)+ ... + 2 + 1$.
And $(n-1)+(n-2)+...+2+1 = \frac{(n-1)((n-1)+1)}{2}$ (Gaussian sum).
Which is equal to $\frac{(n)(n-1)}{2} = \frac{(n^2-n)}{2}$
As such, if G has n vertices, then $\bar{G}$ has $\frac{(n^2-n)}{2} - n$ vertices.
And for G to be self-similar, then we have:
$d = \frac{(n^2-n)}{2} - d$, where d is the number of edges of G.
Hence $d = \frac{(n^2-n)}{4}$.
Since the number of edges d is a natural number, we know that $\frac{(n^2-n)}{4}$ is a natural number. Hence $4|n^2 - n$.
So either $4|n$ or $4|(n-1)$, as if $2|n$, 2 does not divide $(n-1)$ so we know that 4 must divide entirely either one.
Suppose $4|n$. Then $n \equiv 0 \mod 4$.
Suppose $4|n-1$. Then $n \equiv 1 \mod 4$.
Hence for a self-complementary graph with n vertices, either $n \equiv 0 \mod 4$ or $n \equiv 1 \mod 4$.