1. Compute the minimum possible value of 

\[(x-1)^2 + (x-2)^2 + (x-3)^2 + (x-4)^2 + (x-5)^2,\]

for real values of \(x\).

**Answer:** 10

**Solution:** We know that this expression has to be a concave-up parabola (i.e., a parabola that faces upwards), and there is symmetry across the line \(x = 3\). Hence, we conclude that the vertex of the parabola occurs at \(x = 3\). Plugging in, we get \(4+1+0+1+4 = 10\).

2. Express \(\frac{2^3-1}{2^3+1} \times \frac{3^3-1}{3^3+1} \times \frac{4^3-1}{4^3+1} \times \cdots \times \frac{16^3-1}{16^3+1}\) as a fraction in lowest terms.

**Answer:** \(\frac{91}{136}\)

**Solution:** We note 

\[
\prod_{n=2}^{k} \frac{n^3-1}{n^3+1} = \prod_{n=2}^{k} \left(\frac{n-1}{n+1}\right)^2 \left(\frac{n^2+n+1}{n^2-n+1}\right) = \left(\prod_{n=2}^{k} \frac{n-1}{n+1}\right) \left(\prod_{n=2}^{k} \frac{n^2+n+1}{n^2-n+1}\right).
\]

Each product telescopes, yielding \(\frac{1}{k(k+1)} \cdot \frac{k^2+k+1}{3}\). Evaluating at \(k = 16\) yields \(\frac{91}{136}\).

3. If \(x, y,\) and \(z\) are integers satisfying \(xyz + 4(x + y + z) = 2(xy + xz + yz) + 7\), list all possibilities for the ordered triple \((x, y, z)\).

**Answer:** \((1,1,1), (1,3,3), (3,1,3), (3,3,1)\)

**Solution:** Rearranging the given equality yields \(xyz - 2(xy + xz + yz) + 4(x + y + z) - 8 = -1\). But the left side factors as \((x-2)(y-2)(z-2)\). Since all quantities involved are integral, we must have each factor equal to \(\pm 1\). It is easy to verify that the only possibilities for \((x, y, z)\) are those listed.

4. The quartic (4th-degree) polynomial \(P(x)\) satisfies \(P(1) = 0\) and attains its maximum value of 3 at both \(x = 2\) and \(x = 3\). Compute \(P(5)\).

**Answer:** \(-24\)

**Solution:** Consider the polynomial \(Q(x) = P(x) - 3\). \(Q\) has roots at \(x = 2\) and \(x = 3\). Moreover, since these roots are maxima, they both have multiplicity 2. Hence, \(Q\) is of the form \(a(x-2)^2(x-3)^2\), and so \(P(x) = a(x-2)^2(x-3)^2 + 3\). \(P(1) = 0 \implies a = -\frac{3}{4}\), allowing us to compute \(P(5) = -\frac{3}{4}(9)(4) + 3 = -24\).

5. Compute the ordered pair of real numbers \((a,b)\) such that \(a < k < b\) if and only if \(x^3 + \frac{1}{x^3} = k\) does not have a real solution in \(x\).

**Answer:** \((-2, 2)\)

**Solution:** Substitute \(y = x^3\), so now we want to find the values of \(k\) such that \(y + \frac{1}{y} = k\) has no real solutions in \(y\). In particular, since \(y = x^3\) is an invertible function, \(x^3 + \frac{1}{x^3} = k\)
does not have a real solution in $x$ if and only if $y + \frac{1}{y} = k$ has no real solutions in $y$. Therefore, clearing denominators of $y + \frac{1}{y} = k$ gives us $y^2 + 1 = ky$, so $y^2 - ky + 1 = 0$, so this quadratic equation has no solutions when the discriminant is negative. The discriminant is $k^2 - 4$, which is negative when $-2 < k < 2$, so therefore the ordered pair is $\boxed{(-2, 2)}$.

6. If $f$ is a monic cubic polynomial with $f(0) = -64$, and all roots of $f$ are non-negative real numbers, what is the largest possible value of $f(-1)$? (A polynomial is monic if it has a leading coefficient of 1.)

**Answer:** $-125$

**Solution:** If the three roots of $f$ are $r_1, r_2, r_3$, we have $f(x) = x^3 - (r_1 + r_2 + r_3)x^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)x - r_1 r_2 r_3$, so $f(-1) = -1 - (r_1 + r_2 + r_3) - (r_1 r_2 + r_1 r_3 + r_2 r_3) - r_1 r_2 r_3$. Since $r_1 r_2 r_3 = 64$, the arithmetic mean-geometric mean inequality reveals that $r_1 + r_2 + r_3 \geq 3(r_1 r_2 r_3)^{1/3} = 12$ and $r_1 r_2 + r_1 r_3 + r_2 r_3 \geq 3(r_1 r_2 r_3)^{2/3} = 48$. It follows that $f(-1)$ is at most $-1 - 12 - 48 - 64 = \boxed{-125}$. We have equality when all roots are equal, i.e. $f(x) = (x - 4)^3$.

7. There exist two triples of real numbers $(a, b, c)$ such that $a - \frac{1}{b}$, $b - \frac{1}{c}$, and $c - \frac{1}{a}$ are the roots to the cubic equation $x^3 - 5x^2 - 15x + 3$ listed in increasing order. Denote those $(a_1, b_1, c_1)$ and $(a_2, b_2, c_2)$. If $a_1$, $b_1$, and $c_1$ are the roots to monic cubic polynomial $f$ and $a_2$, $b_2$, and $c_2$ are the roots to monic cubic polynomial $g$, find $f(0)^3 + g(0)^3$.

**Answer:** $-14$

**Solution:** By Viétas Formulas, we have that $f(0) = -a_1 b_1 c_1$ and $g(0) = -a_2 b_2 c_2$. Additionally, $(a - \frac{1}{b})(b - \frac{1}{c})(c - \frac{1}{a}) = -3$ and $(a - \frac{1}{b}) + (b - \frac{1}{c}) + (c - \frac{1}{a}) = 5$. Expanding the first expression yields $-3 = abc - \frac{1}{abc} - ((a + b + c) - (\frac{1}{a} + \frac{1}{b} + \frac{1}{c})) = abc - \frac{1}{abc} - 5$. This is equivalent to $(abc)^2 - 2(abc) - 1 = 0$, so $abc = 1 \pm \sqrt{2}$. It follows that $f(0)^3 + g(0)^3 = -(1 + \sqrt{2})^3 - (1 - \sqrt{2})^3 = \boxed{-14}$.

8. How many positive integers $n$ are there such that for any natural numbers $a, b$, we have $n \mid (a^2b + 1)$ implies $n \mid (a^2 + b)$? (Note: The symbol $\mid$ means “divides”; if $x \mid y$ then $y$ is a multiple of $x$.)

**Answer:** 20

**Solution:** Let $\mathcal{P}$ represent the property of $n$ such that $n \mid a^2b + 1 \Rightarrow n \mid a^2 + b$ for all $a, b \in \mathbb{N}$. Let $\mathcal{Q}$ represent the property of $n$ such that $(a, n) = 1 \Rightarrow n \mid a^4 - 1$ for all $a \in \mathbb{N}$. We shall prove that they are equivalent.

Proof that $\mathcal{P} \Rightarrow \mathcal{Q}$: Let $a$ be a positive integer with $(a, n) = 1$. By Bézout’s identity, we can find $b \in \mathbb{N}$ such that $n \mid a^2b + 1$. By $\mathcal{P}$, $n \mid a^2 + b$. Then $a^4 - 1 = a^2(a^2 + b) - (a^2b + 1)$, so $n \mid a^4 - 1$.

Proof that $\mathcal{Q} \Rightarrow \mathcal{P}$: Let $a, b$ be positive integers with $n \mid a^2b + 1$. Clearly $(a, n) = 1$, so $n \mid a^4 - 1$. Then $a^2(a^2 + b) = (a^4 - 1) + (a^2b + 1)$. Since $a$ and $n$ are relatively prime, $n \mid a^2 + b$. 
Now we wish to find all $n$ with property $\mathcal{Q}$. If $a$ is odd, we have $a^4 - 1 = (a^2 - 1)(a^2 + 1)$, $a^2 \equiv 1 \pmod{8}$, and $a^2 + 1$ is even, so $16 \mid a^4 - 1$. If $(a, 3) = 1$, we have $a^2 \equiv 1 \pmod{3}$, so $3 \mid a^4 - 1$. If $(a, 5) = 1$, we have $5 \mid a^4 - 1$ by Fermat’s Little Theorem. This argument shows that $n \mid 240$ is sufficient.

To show $n \mid 240$ is necessary, suppose $n$ has property $\mathcal{Q}$, and let $n = 2^a \cdot k$, where $k$ is odd. If $k > 1$, then $(k - 2, n) = 1$, so by $\mathcal{Q}$ we conclude that $n \mid (k - 2)^4 - 1$. Then $k \mid (k - 2)^4 - 1$, but $(k - 2)^4 \equiv (-2)^4 \equiv 16 \pmod{k}$, so $k \mid 15$. Now, since $(11, n) = 1$, $n \mid 11^4 - 1$, so $2^a \mid 11^4 - 1$, resulting in $a \leq 4$. Thus $n \mid 240$ is also necessary.

The number of natural numbers $n$ such that property $\mathcal{P}$ holds is simply the number of positive integer divisors of 240, which is $(4 + 1)(1 + 1)(1 + 1) = 20$.

9. The function $f(x)$ is known to be of the form $\prod_{i=1}^{n} f_i(a_i x)$, where $a_i$ is a real number and $f_i(x)$ is either $\sin(x)$ or $\cos(x)$ for $i = 1, \ldots, n$. Additionally, $f(x)$ is known to have zeros at every integer between 1 and 2012 (inclusive) except for one integer $b$. Find the sum of all possible values of $b$.

Answer: 2047

Solution: The possible values of $b$ are the powers of two not exceeding 2012 (including $2^0 = 1$). The following proof uses the fact that the zeroes of sine and cosine are precisely numbers of the form $t \pi$ and $(t + 1/2) \pi$, respectively, for $t$ an integer.

Suppose $b$ is not a power of 2. Then it can be written as $2^m(1 + 2k)$ for $m \geq 0$, $k > 0$. Since $2^m < b$, by assumption one of the $f_i$ must have a root at $2^m$. But then this same $f_i$ must have a root at $b$:

- If $f_i(x) = \sin(ax)$ and $f_i(2^m) = 0$, then $2^m a = t \pi$ for some integer $t$, so $f_i(b) = \sin(ba) = \sin((1 + 2k)2^m a) = \sin((1 + 2k)t \pi) = 0$.
- If $f_i(x) = \cos(ax)$ and $f_i(2^m) = 0$, then $2^m a = (t + 1/2) \pi$ for some integer $t$ so $f_i(b) = \cos(ba) = \cos((1 + 2k)2^m a) = \cos((1 + 2k)(t + 1/2) \pi) = \cos((t + k + 2kt + 1/2) \pi) = 0$

This is a contradiction, so $b$ must be a power of 2.

For each $b$ of the form $2^n$, we can construct an $f$ that works by using cosine terms to cover integers preceding $b$ and sine terms thereafter:

$$f(x) = \left( \prod_{i=1}^{m} \cos(\pi x / 2^i) \right) \left( \prod_{j=b+1}^{2012} \sin(\pi x / j) \right)$$

has a root at every positive integer at most 2012 except $b$.

Hence, our final answer is $1 + 2 + 4 + \ldots + 1024 = 2048 - 1 = 2047$.

10. For real numbers $(x, y, z)$ satisfying the following equations, find all possible values of $x + y + z$.

$$x^2 y + y^2 z + z^2 x = -1$$
$$x y^2 + y z^2 + z x^2 = 5$$
$$x y z = -2$$
Answer: \(2 \text{ or } \sqrt[3]{\frac{1}{2}}\)

Solution: Let \(x/y = a\), \(y/z = b\), and \(z/x = c\). Then \(abc = 1\). By dividing the first two equations by the third equation, we have \(a + b + c = -5/2\) and \(1/a + 1/b + 1/c = ab + bc + ca = 1/2\). So \(a, b, c\) are roots of \(2X^3 + 5X^2 + X - 2 = 0\). By observation, the three roots of this equation are \(-2, -1, 1/2\). Without loss of generality, assume that \(a = -1\) and \(y = -x\). If \(c = -2\), then we have \(z = -2x\), so \(xyz = 2x^3 = -2\), and thus \(x = -1\). In this case we have

\[(x, y, z) = (-1, 1, 2).\]

If \(c = 1/2\), then \(z = x/2\), so \(xyz = -x^3/2 = -2\), or \(x = \sqrt[3]{4}\). In this case we have

\[(x, y, z) = \left(\sqrt[3]{4}, -\sqrt[3]{4}, \frac{3}{\sqrt[3]{2}}\right).\]

It follows that \(x + y + z\) can be either \(2\) or \(\sqrt[3]{\frac{1}{2}}\).