1. Compute the minimum possible value of

$$(x-1)^2+(x-2)^2+(x-3)^2+(x-4)^2+(x-5)^2,$$

for real values of x.

Answer: 10

Solution: We know that this expression has to be a concave-up parabola (i.e. a parabola that faces upwards), and there is symmetry across the line $x=3$. Hence, we conclude that the vertex of the parabola occurs at $x=3$. Plugging in, we get $4+1+0+1+4=\boxed{10}$.

2. Express $\frac{2^3-1}{2^3+1}\times\frac{3^3-1}{3^3+1}\times\frac{4^3-1}{4^3+1}\times\cdots\times\frac{16^3-1}{16^3+1}$ as a fraction in lowest terms.

Answer: $\mathbf{\frac{91}{136}}$

Solution: We note

$$\prod_{n=2}^{k}\frac{n^3-1}{n^3+1}=\prod_{n=2}^{k}\frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)}=\left(\prod_{n=2}^{k}\frac{n-1}{n+1}\right)\left(\prod_{n=2}^{k}\frac{n^2+n+1}{n^2-n+1}\right).$$

Each product telescopes, yielding $\frac{1\cdot 2}{k\cdot(k+1)}\cdot\frac{k^2+k+1}{3}$. Evaluating at $k=16$ yields $\boxed{\dfrac{91}{136}}$.

3. If $x,y,$ and z are integers satisfying $xyz+4(x+y+z)=2(xy+xz+yz)+7$, list all possibilities for the ordered triple (x,y,z).

Answer: $\mathbf{(1,1,1),(1,3,3),(3,1,3),(3,3,1)}$

Solution: Rearranging the given equality yields $xyz-2(xy+xz+yz)+4(x+y+z)-8=-1$. But the left side factors as $(x-2)(y-2)(z-2)$. Since all quantities involved are integral, we must have each factor equal to ± 1. It is easy to verify that the only possibilities for (x,y,z) are those listed.

4. The quartic (4th-degree) polynomial $P(x)$ satisfies $P(1)=0$ and attains its maximum value of 3 at both $x=2$ and $x=3$. Compute $P(5)$.

Answer: $\mathbf{-24}$

Solution: Consider the polynomial $Q(x)=P(x)-3$. Q has roots at $x=2$ and $x=3$. Moreover, since these roots are maxima, they both have multiplicity 2. Hence, Q is of the form $a(x-2)^2(x-3)^2$, and so $P(x)=a(x-2)^2(x-3)^2+3$. $P(1)=0 \implies a=-\frac{3}{4}$, allowing us to compute $P(5)=-\frac{3}{4}(9)(4)+3=\boxed{-24}$.

5. Compute the ordered pair of real numbers (a,b) such that $a<k<b$ if and only if $x^3+\dfrac{1}{x^3}=k$ does not have a real solution in x.

Answer: $\mathbf{(-2,2)}$

Solution: Substitute $y=x^3$, so now we want to find the values of k such that $y+\dfrac{1}{y}=k$ has no real solutions in y. In particular, since $y=x^3$ is an invertible function, $x^3+\dfrac{1}{x^3}=k$

does not have a real solution in x if and only if $y + \dfrac{1}{y} = k$ has no real solutions in y. Therefore, clearing denominators of $y+\dfrac{1}{y} = k$ gives us $y^2+1 = ky$, so $y^2-ky+1 = 0$, so this quadratic equation has no solutions when the discriminant is negative. The discriminant is $k^2 - 4$, which is negative when $-2 < k < 2$, so therefore the ordered pair is $\boxed{(-2, 2)}$.

6. If f is a monic cubic polynomial with $f(0) = -64$, and all roots of f are non-negative real numbers, what is the largest possible value of $f(-1)$? (A polynomial is monic if it has a leading coefficient of 1.)

   Answer: -125

   Solution: If the three roots of f are r_1, r_2, r_3, we have $f(x) = x^3-(r_1+r_2+r_3)x^2+(r_1r_2+r_1r_3+r_2r_3)x-r_1r_2r_3$, so $f(-1) = -1-(r_1+r_2+r_3)-(r_1r_2+r_1r_3+r_2r_3)-r_1r_2r_3$. Since $r_1r_2r_3 = 64$, the arithmetic mean-geometric mean inequality reveals that $r_1 + r_2 + r_3 \geq 3(r_1r_2r_3)^{1/3} = 12$ and $r_1r_2 + r_1r_3 + r_2r_3 \geq 3(r_1r_2r_3)^{2/3} = 48$. It follows that $f(-1)$ is at most $-1 - 12 - 48 - 64 = \boxed{-125}$. We have equality when all roots are equal, i.e. $f(x) = (x - 4)^3$.

7. There exist two triples of real numbers (a, b, c) such that $a - \frac{1}{b}$, $b - \frac{1}{c}$, and $c - \frac{1}{a}$ are the roots to the cubic equation $x^3 - 5x^2 - 15x + 3$ listed in increasing order. Denote those (a_1, b_1, c_1) and (a_2, b_2, c_2). If a_1, b_1, and c_1 are the roots to monic cubic polynomial f and a_2, b_2, and c_2 are the roots to monic cubic polynomial g, find $f(0)^3 + g(0)^3$.

   Answer: -14

   Solution: By Viéta's Formulas, we have that $f(0) = -a_1b_1c_1$ and $g(0) = -a_2b_2c_2$. Additionally, $(a - \frac{1}{b})(b - \frac{1}{c})(c - \frac{1}{a}) = -3$ and $(a - \frac{1}{b}) + (b - \frac{1}{c}) + (c - \frac{1}{a}) = 5$. Expanding the first expression yields $-3 = abc - \frac{1}{abc} - ((a+b+c) - (\frac{1}{a} + \frac{1}{b} + \frac{1}{c})) = abc - \frac{1}{abc} - 5$. This is equivalent to $(abc)^2 - 2(abc) - 1 = 0$, so $abc = 1 \pm \sqrt{2}$. It follows that $f(0)^3 + g(0)^3 = -(1 + \sqrt{2})^3 - (1 - \sqrt{2})^3 = \boxed{-14}$.

8. How many positive integers n are there such that for any natural numbers a, b, we have $n \mid (a^2b + 1)$ implies $n \mid (a^2 + b)$? (Note: The symbol $\mid$ means "divides"; if $x \mid y$ then y is a multiple of x.)

   Answer: 20

   Solution: Let $\mathcal{P}$ represent the property of n such that $n \mid a^2b + 1 \Rightarrow n \mid a^2 + b$ for all $a, b \in \mathbb{N}$. Let $\mathcal{Q}$ represent the property of n such that $(a, n) = 1 \Rightarrow n \mid a^4 - 1$ for all $a \in \mathbb{N}$. We shall prove that they are equivalent.

   Proof that $\mathcal{P} \Rightarrow \mathcal{Q}$: Let a be a positive integer with $(a, n) = 1$. By Bézout's identity, we can find $b \in \mathbb{N}$ such that $n \mid a^2b+1$. By $\mathcal{P}$, $n \mid a^2+b$. Then $a^4-1 = a^2(a^2+b)-(a^2b+1)$, so $n \mid a^4 - 1$.

   Proof that $\mathcal{Q} \Rightarrow \mathcal{P}$: Let a, b be positive integers with $n \mid a^2b + 1$. Clearly $(a, n) = 1$, so $n \mid a^4 - 1$. Then $a^2(a^2 + b) = (a^4 - 1) + (a^2b + 1)$. Since a and n are relatively prime, $n \mid a^2 + b$.

Now we wish to find all n with property $\mathcal{Q}$. If a is odd, we have $a^4-1=(a^2-1)(a^2+1)$, $a^2 \equiv 1 \pmod 8$, and a^2+1 is even, so $16 \mid a^4-1$. If $(a,3)=1$, we have $a^2 \equiv 1 \pmod 3$, so $3 \mid a^4-1$. If $(a,5)=1$, we have $5 \mid a^4-1$ by Fermat's Little Theorem. This argument shows that $n \mid 240$ is sufficient.

To show $n \mid 240$ is necessary, suppose n has property $\mathcal{Q}$, and let $n=2^a \cdot k$, where k is odd. If $k>1$, then $(k-2,n)=1$, so by $\mathcal{Q}$ we conclude that $n \mid (k-2)^4-1$. Then $k \mid (k-2)^4-1$, but $(k-2)^4 \equiv (-2)^4 \equiv 16 \pmod k$, so $k \mid 15$. Now, since $(11,n)=1$, $n \mid 11^4-1$, so $2^a \mid 11^4-1$, resulting in $a \le 4$. Thus $n \mid 240$ is also necessary.

The number of natural numbers n such that property $\mathcal{P}$ holds is simply the number of positive integer divisors of 240, which is $(4+1)(1+1)(1+1)=\boxed{20}$.

9. The function $f(x)$ is known to be of the form $\prod_{i=1}^{n} f_i(a_i x)$, where a_i is a real number and $f_i(x)$ is either $\sin(x)$ or $\cos(x)$ for $i=1,\dots,n$. Additionally, $f(x)$ is known to have zeros at every integer between 1 and 2012 (inclusive) except for one integer b. Find the sum of all possible values of b.

Answer: 2047

Solution: The possible values of b are the powers of two not exceeding 2012 (including $2^0=1$). The following proof uses the fact that the zeroes of sine and cosine are precisely numbers of the form $t\pi$ and $(t+1/2)\pi$, respectively, for t an integer.

Suppose b is not a power of 2. Then it can be written as $2^m(1+2k)$ for $m \ge 0$, $k>0$. Since $2^m<b$, by assumption one of the f_i must have a root at 2^m. But then this same f_i must have a root at b:

- If $f_i(x)=\sin(ax)$ and $f_i(2^m)=0$, then $2^m a=t\pi$ for some integer t, so

$$f_i(b)=\sin(ba)=\sin((1+2k)2^m a)=\sin((1+2k)t\pi)=0.$$

- If $f_i(x)=\cos(ax)$ and $f_i(2^m)=0$, then $2^m a=(t+1/2)\pi$ for some integer t so

$$f_i(b)=\cos(ba)=\cos((1+2k)2^m a)=\cos((1+2k)(t+1/2)\pi)=\cos((t+k+2kt+1/2)\pi)=0$$

This is a contradiction, so b must be a power of 2.

For each b of the form 2^m, we can construct an f that works by using cosine terms to cover integers preceding b and sine terms thereafter:

$$f(x)=\left(\prod_{i=1}^{m}\cos(\pi x/2^i)\right)\left(\prod_{j=b+1}^{2012}\sin(\pi x/j)\right)$$

has a root at every positive integer at most 2012 except b.

Hence, our final answer is $1+2+4+\dots+1024=2048-1=\boxed{2047}$.

10. For real numbers (x,y,z) satisfying the following equations, find all possible values of $x+y+z$.

$$\begin{aligned} x^2y+y^2z+z^2x &= -1 \\ xy^2+yz^2+zx^2 &= 5 \\ xyz &= -2 \end{aligned}$$

Answer: 2 or $\sqrt[3]{\frac{1}{2}}$

Solution: Let $x/y = a$, $y/z = b$, and $z/x = c$. Then $abc = 1$. By dividing the first two equations by the third equation, we have $a + b + c = -5/2$ and $1/a + 1/b + 1/c = ab + bc + ca = 1/2$. So a, b, c are roots of $2X^3 + 5X^2 + X - 2 = 0$. By observation, the three roots of this equation are $-2, -1$, and $1/2$. Without loss of generality, assume that $a = -1$ and $y = -x$. If $c = -2$, then we have $z = -2x$, so $xyz = 2x^3 = -2$, and thus $x = -1$. In this case we have

$$(x, y, z) = (-1, 1, 2).$$

If $c = 1/2$, then $z = x/2$, so $xyz = -x^3/2 = -2$, or $x = \sqrt[3]{4}$. In this case we have

$$(x, y, z) = \left(\sqrt[3]{4}, -\sqrt[3]{4}, \sqrt[3]{\frac{1}{2}}\right).$$

It follows that $x + y + z$ can be either $\boxed{2 \text{ or } \sqrt[3]{\frac{1}{2}}}$.