# $12^{\text{th}}$ Annual Johns Hopkins Math Tournament
## Saturday, February 19, 2011
## Power Round-Poles and Polars

### 1. Definition and Basic Properties

In the cartesian plane, the *polar* of the point $P = (p_x, p_y)$ with respect to the unit circle is the line $p_x x + p_y y = 1$. The *pole* of the line $l : l_x x + l_y y = 1$ with respect to the unit circle is the point $(l_x, l_y)$.

1. (**12 = 2 × 6**) Graph the polars (or the poles) of the following figures, together with the original figures (separate graph for each part). Also draw your coordinate axes. You don't need to justify.
   (1) $A(1, 1)$ (the "A" is just a label for the point $(1, 1)$)
   (2) $b : y = x + 2$
   (3) $C(-0.6, 0.8)$
   (4) $d : x = 2y + 2$
   (5) $E$ : the pole of line $a$, the polar of $A$
   (6) $f$ : the polar of the intersection of lines $a$ and $b$
2. (**12**) How many intersections are there between the polar of $P$ and the unit circle, given that $P$ is (1) inside, (2) on, and (3) outside the unit circle? Justify.

The *reciprocation* transform takes a point which is not the origin to its polar, and a line which do not go through the origin to its pole. It is easy to see that reciprocation establishes a bijection between non-origin points and lines which do not go through the origin, and it is its own inverse: $l$ is the polar of the point $P$ if and only if $P$ is the pole of the line $l$.

3. (**12**) The more common definitions of poles and polars are the following:
   Define the *inversion* of the nonorigin point $P$ with respect to the unit circle as follows: it is the point $P'$ on the ray $\overline{OP}$ satisfying $OP \cdot OP' = 1$. Then define the polar of $P$ to be the line going through $P'$ which is perpendicular to $OP$. Also for a line $l$ which does not go through the origin, define the pole of $l$ to be the inversion of the foot of the perpendicular from $O$ to $l$.
   Show that this definition is equivalent to our initial definition.
4. (**12**) Generalize the concept of reciprocation to reciprocation around any circle $(x-a)^2 + (y-b)^2 = r^2$ (think of the reciprocation we have defined as reciprocation around the unit circle). Find the equation of the polar of $(p, q)$ (where $(p, q) \neq (a, b)$) and the coordinates of the pole of $c(x-a) + d(y-b) = 1$ (where $(c, d) \neq (0, 0)$). Your definition should be consistent with the rescaling and translation, but you don't need to justify your answer.

### 2. The Duality Principle

For the remaining problems you don't need to use generalized reciprocation unless noted.

5. (**6**) Prove the following: $A$ is on the polar of $B$ if and only if $B$ is on the polar of $A$.
6. (**12**) The following are corollaries of 5. Let $A, B, C$ be non-origin points, and $a, b, c$ be their respective polars. Prove the following:
   (a) The point $A$ is on $b$ if and only if the point $B$ is on $a$.
   (b) The pole of $AB$ is the intersection of $a$ and $b$. Conversely, the polar of the intersection of $a$ and $b$ is $AB$.
   (c) Points $A, B, C$ are collinear (on the same line) if and only if lines $a, b, c$ are concurrent (go through the same point).

In geometry, *duality* refers to geometric transformations that replace points by lines and lines by points while preserving *incidence properties*, the relation of a line going through a point. It is easy to see that reciprocation is a duality for nonorigin points and lines not going through the origin. This leads to a general principle called the *duality principle*: any theorem about incidences between points and lines may be transformed into another theorem about lines and points, by a substitution of the appropriate words. (Wikipedia.) The transformed theorem is sometimes called the *dual theorem* or *reciprocal*.

7. (**6**) State the dual theorem of the following.
   (Pappus's theorem) Given $A, B, C$ collinear and $D, E, F$ collinear (not necessarily in that order), the three intersection points $X = BF \cap CE$, $Y = AF \cap CD$, and $Z = AE \cap BD$ are also collinear.
8. (**12**) Explain how to generalize the duality principle to theorems which also include incidences between a given circle and some points. Use this generalization to state the dual theorem of the following. Your generalized reciprocation from problem 4 will be useful.
   (Pascal's theorem) For a cyclic hexagon $ABCDEF$ (not necessarily around the circle in that order), the three intersections of opposite sides $X = AB \cap DE$, $Y = BC \cap EF$, $Z = CD \cap FA$ are collinear.

## 3. Reciprocation and Cyclic Quadrilaterals

9. (**24 = 4 × 6**) Given a circle, let $P$ be a point inside the circle, and $AC$ and $BD$ be two chords of the circle which go through $P$. Let $Q$ be the intersection of the lines $AD$ and $BC$ (outside the circle). Then $Q$ is on the polar of $P$. Prove this statement following the steps below:
   (1) Let $X$ be the point on the segment $AD$ satisfying $AX/XD = AQ/QD$. Supposing that the circle is the unit circle, points $A$ and $D$ have coordinates $A = (x_a, y_a)$ and $D = (x_d, y_d)$, and the ratio $AQ : QD = m : n$ (where $m, n \neq 0$), express the coordinates of $Q$ and $X$ in terms of $x_a, y_a, x_d, y_d, m, n$. We say that $X$ is the *harmonic conjugate* of the three points $A, D, Q$.
   (2) Show that $X$ is on the polar of $Q$.
   (3) Define $Y$ be the harmonic conjugate of $B, C, Q$. Show that $AC, XY, BD$ intersect at one point. To show this, let $P_1$ and $P_2$ be the intersection of $AC$ and $XY$, $BD$ and $XY$ respectively, and calculate the ratio $P_1X/P_1Y$ and $P_2X/P_2Y$ using Menelaus' theorem (if $ABC$ is a triangle and $P, Q, R$ are points on the lines $AB$, $BC$, and $CA$ respectively, then $P, Q, R$ are collinear if and only if $\frac{AP}{PB}\frac{BQ}{QC}\frac{CR}{RA} = -1$) to see that those two coincide.
   (4) Conclude that $P$ is on the polar of $Q$.
10. (**12**) For cyclic quadrilateral $ABCD$, we can consider three points $P = AC \cap BD$, $Q = AD \cap BC$ and $R = AB \cap CD$. Show that the polar of each point goes through other two ponts. What can be said about the orthocenter of $PQR$?
11. (**12**) Prove the following "six points on the line" property: for cyclic quadrilateral $ABCD$ we have (1) the intersection of $AC$ and $BD$, (2) the intersection of $AD$ and $BC$, (3) the intersection of the tangent at $A$ and the tangent at $B$, (4) the intersection of the tangent at $C$ and the tangent at $D$, and (5) and (6) the tangent points of the two tangents from $AB \cap CD$ to the circle. The property is that all of these points are on the same line.
12. (**12**) This is not related to poles and polars, but can you prove that (7) the intersection of the circumcircles of $ADP$ and $BCP$ (other than $P$) is also on the line?

## 4. Self-Polar Triangles

Given points $A$ and $B$ and their polars $a$ and $b$ such that $A$ is on $b$ and $B$ is on $a$, we say that $A$ and $B$ are conjugate points and that $a$ and $b$ are conjugate lines.

13. (**6**) Let $A$ be inside a circle with center $O$ and $B$ be outside the circle so that $A$ and $B$ are conjugate points. Let $C$ be the intersection of $A$ and $B$. Show the following three properties of the triangle $ABC$: (1) each vertex is the pole of the opposite side, (2) any two vertices are conjugate points, and (3) any two sides are conjugate lines. $ABC$ is called a *self-polar triangle.*
14. (**10**) Let $ABC$ be any triangle with $\angle A$ obtuse. Find the center and radius of the unique circle with respect to which $ABC$ is self-polar. This is called the *polar circle* of $ABC$.
15. (**10**) Let $ABC$ be as in part 13. Find the locus of those points which are inversions of points on the circumcircle of $ABC$ with respect to its polar circle (inversion is defined in problem 3).

## 5. Counting

Taking the dual of certain counting problems can make the counting easier. Apply the idea of duality to help solve the following problem.

16. (**30**) For any $n > 0$, take $n$ distinguishable points $P_1, ..., P_n$ in the plane. For any line $\ell$ not going through any of the points, let the "linear partition of $P_1, ..., P_n$ by $\ell$", which we denote $M_\ell(P_1, ..., P_n)$, be the unordered pair $\{L, R\}$ where $L$ is the set of points on one side of the line and $R$ is the set of points on the other side of the line. Let $C(P_1, ..., P_n)$ be the number of distinct unordered pairs $M_\ell(P_1, ..., P_n)$ as $\ell$ ranges over all lines. i.e., $C(P_1, ..., P_n)$ is the number of different ways we can split $P_1, ..., P_n$ with lines.

Here are some examples. For one point, $C = 1$ because there is only one way to "split" the set of one point. For two points, $C = 2$ because we can have both points on one side of the line or each point on an opposite side of the line. For three collinear points $P_1, P_2, P_3$, $C = 3$ because the linear partitions are $\mid P_1, P_2, P_3$ and $P_1 \mid P_2, P_3$ and $P_1, P_2 \mid P_3$. For three points on the vertices of an equilateral triangle, $C = 4$ because we can have all the points on the same side of the line or we can split off any one of the three points from the other two.

What is the maximum of $C(P_1, ..., P_n)$ over all choices of $n$ points? Prove that your answer is the maximum. Duality will be helpful.