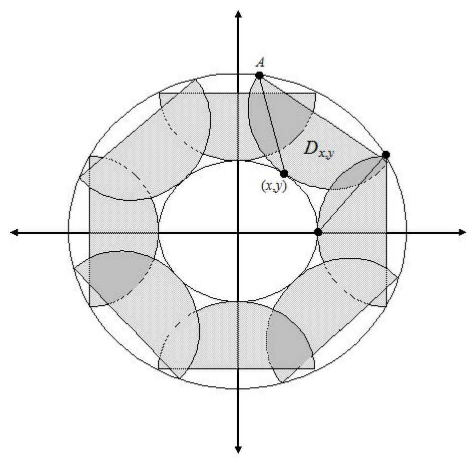
12^{th} Annual Johns Hopkins Math Tournament
Saturday, February 19, 2011

Geometry Subject Test

1. [**1025**] Let $D_{x,y}$ denote the half-disk of radius 1 with its curved boundary externally tangent to the unit circle at the point (x, y), such that the straight boundary of the disk is parallel to the tangent line (so the point of tangency is the middle of the curved boundary). Find the area of the union of the $D_{x,y}$ over all (x, y) with $x^2 + y^2 = 1$ (that is, (x, y) is on the unit circle).
Answer: $\boxed{4\pi}$ Given any point of tangency (x, y), the points on the circle farthest away are the two diametrically opposite points on the straight boundary. The length of the segment connecting (x, y) and either of these two points (pick one and call it A) is $\sqrt{2}$ because the half-disk has radius 1; the line segment can be viewed as the hypotenuse of the right triangle spanned by two radii of $D_{x,y}$.



As the union of the $D_{x,y}$ sweep around the unit circle, the segment from (x, y) to A sweeps out an annulus of width $\sqrt{2}$; that is, it sweeps out a large circle of radius $\sqrt{5}$ with an inner circle of radius 1 removed. Hence the area of the remaining region is difference between the areas of the two circles: $5\pi - \pi = 4\pi$.

2. [**1026**] Let circle O have radius 5 with diameter AE. Point F is outside circle O such that lines AF and EF intersect circle O at points B, and D, respectively. If $AF = 10$ and $m\angle FAE = 30°$, then the perimeter of quadrilateral $ABDE$ can be expressed as $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, where a, b, c, d are rational. Find $a + b + c + d$.
Answer: $\boxed{15}$ After some angle chasing, we find that $m\angle DBF = m\angle DFB = 75°$, which implies that $DF = DB$. Hence the desired perimeter is equal to $AF - BF + AE + FE = 20 - BF + FE$. By the Law of Sines,

$$\frac{FE}{\sin 30°} = \frac{10}{\sin 75°} \Rightarrow FE = \frac{5}{\frac{\sqrt{6}+\sqrt{2}}{4}} = 5\sqrt{6} - 5\sqrt{2}.$$

Now, to find BF draw the altitude from O to AB intersecting AB at P. This forms a $30 - 60 - 90$ triangle, so we can see that

$$AP = 5\frac{\sqrt{3}}{2} = \frac{10 - BF}{2} \Rightarrow BF = 10 - 5\sqrt{3}.$$

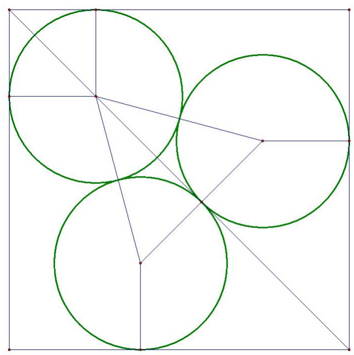
Hence the answer is $20 + (5\sqrt{6} - 5\sqrt{2}) - (10 - 5\sqrt{3}) = 10 - 5\sqrt{2} + 5\sqrt{3} + 5\sqrt{6}$, so the desired sum is $10 - 5 + 5 + 5 = 15$.

3. [**1028**] In a unit cube $ABCD - EFGH$, an equilateral triangle BDG cuts out a circle from the circumsphere of the cube. Find the area of the circle.
Answer: $\boxed{\frac{\pi}{6}}$ Consider the cube of side length 2 and divide the answer by 4 later. Set the coordinates

of the vertices of the cube to be $(\pm 1, \pm 1, \pm 1)$. Then the plane going through a equilateral triangle can be described as $x+y+z=1$. The distance to the plane from the origin is $\frac{1}{\sqrt{3}}$, as $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is the foot of the perpendicular from $(0,0,0)$. Thus the radius of the circle is $\sqrt{1-\left(\frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{2}{3}}$, so the area is $\frac{2}{3}\pi$. In the case of the unit cube we should divide it by 4 to get the answer $\frac{\pi}{6}$.

4. [**1032**] Compute the largest value of r such that three non-overlapping circles of radius r can be inscribed in a unit square.

   Answer: $\boxed{\dfrac{\sqrt{2}}{1+2\sqrt{2}+\sqrt{3}}}$ The three circles will be inscribed in such a way that one altitude of the equilateral triangle formed by the centers of the three circles will coincide with a diagonal of the square, as in the figure below. Indeed, by the Pigeonhole Principle, one circle must lie tangent to two sides of the square, and in any orientation other than the one below, the circles can be dilated.



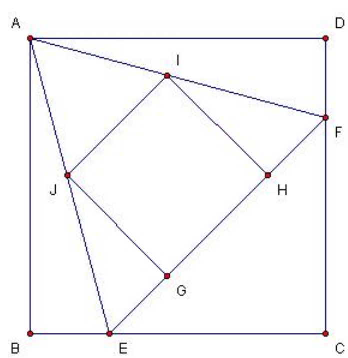
   Label the square $ABCD$ starting in the upper left and going clockwise. The line from the center of the top left circle to A has length $\sqrt{2}r$, and the equilateral triangle formed by the radii has height $\sqrt{3}r$. Then the line from the base of the equilateral triangle to C has length $\sqrt{2}-(\sqrt{2}+\sqrt{3})r$. Now, draw lines from the centers of the two lower circles to C to form four triangles. Observe that these triangles are identical, with angle $\frac{\pi}{8}$ in the lower right. Then

$$\tan\left(\frac{\pi}{8}\right) = \sqrt{2}-1 = \frac{r}{\sqrt{2}-(\sqrt{2}+\sqrt{3})r}$$

   and solving for r gives the desired answer, or some equivalent expression.

5. [**1040**] Let $ABCD$ be a unit square. Point E is on BC, point F is on DC, $\triangle AEF$ is equilateral, and $GHIJ$ is a square in $\triangle AEF$ such that GH is on EF. Compute the area of square $GHIJ$.

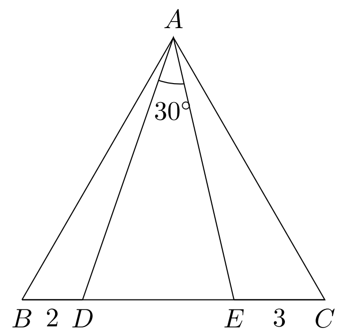
   Answer: $\boxed{312-180\sqrt{3}}$ The setup is as follows:



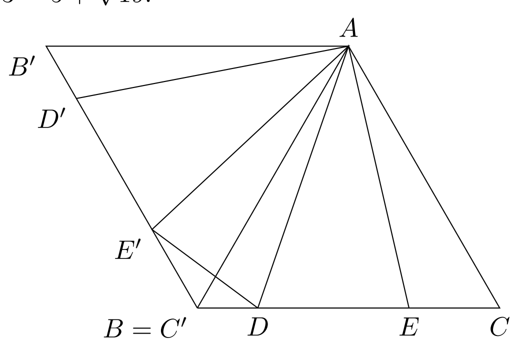
   First let a be the length of AE. Then $CE = a/\sqrt{2}$, $BE = 1 - a/\sqrt{2}$ so $AE^2 = a^2 = 1 + BE^2 = 2 - \sqrt{2}a + a^2/2$. Solving it gives $a^2 + 2\sqrt{2}a - 4 = 0$, $(a+\sqrt{2})^2 = 6$ so $a = \sqrt{6}-\sqrt{2}$. Next let b

be the length of IJ. Then AIJ is equilateral so $AJ = b$. Also $JE = 2/\sqrt{3}b$, so $AE = a = \frac{2+\sqrt{3}}{\sqrt{3}}b$, $b = (2-\sqrt{3})(\sqrt{3})(\sqrt{6}-\sqrt{2}) = \sqrt{2}(9-5\sqrt{3})$. Squaring it gives $312 - 180\sqrt{3}$.

6. [**1056**] Let $\triangle ABC$ be equilateral. Two points D and E are on side BC (with order B, D, E, C), and satisfy $\angle DAE = 30°$. If $BD = 2$ and $CE = 3$, what is BC?

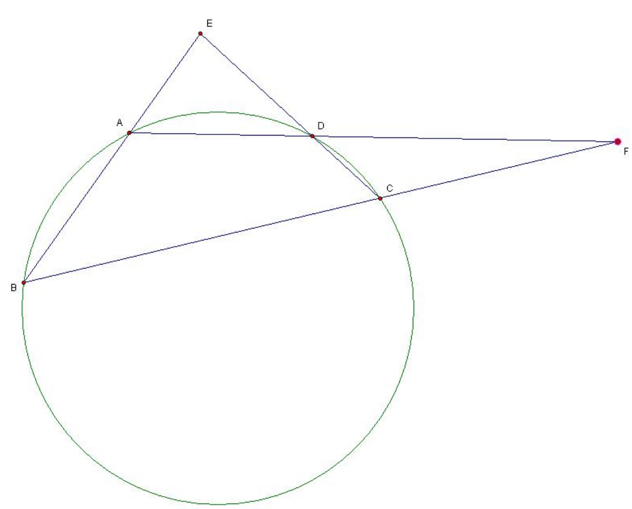


Answer: $\boxed{5+\sqrt{19}}$ Rotate the figure around A by $60°$ so that C comes at the place of B. Let B', C', D', E' be corresponding points of the moved figure. Since $\angle E'AD = \angle E'AC' + \angle C'AD = \angle EAC + \angle BAD = 30° = \angle EAD$, $E'A = EA$ and $DA = D'A$, one has $E'D = ED$. So $BC = BD + DE + EA$ can be found out if we know $E'D$. But $E'D = \sqrt{E'B^2 + BD^2 - 2 \cdot E'B \cdot BD \cdot \cos 120°} = \sqrt{19}$, so $BC = 2 + \sqrt{19} + 3 = 5 + \sqrt{19}$.



7. [**1088**] Let $ABCD$ be cyclic quadrilateral with $AB = 6$, $BC = 12$, $CD = 3$, and $DA = 6$. Let E, F be the intersection of lines AB and CD, lines AD and BC, respectively. Find EF.

Answer: $\boxed{10\sqrt{2}}$ We have $\triangle ADE \sim \triangle CBE$, and ratio is $AD : CB = 1 : 2$. Let $AE = p$ and $DE = q$, then we have $AB = BE - AE = 2DE - AE = 2q - p$ and $CD = 2p - q$. Solving for p and q we have $p = 4$ and $q = 5$. Similarly we have $FC = 8$ and $FD = 10$.



Let $\angle B = \theta$, then $\angle FDE = \pi - \theta$. Apply the cosine law to $\triangle EBF$ to get

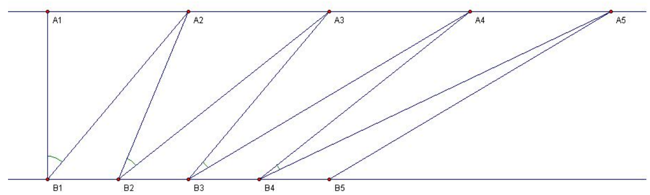
$$EF^2 = BE^2 + BF^2 - 2BE \cdot BF \cdot \cos\theta = 10^2 + 20^2 - 2 \cdot 10 \cdot 20 \cos\theta = 500 - 400\cos\theta$$

and to $\triangle EDF$ to get

$$EF^2 = DE^2 + DF^2 + 2 \cdot DE \cdot DF \cos\theta = 5^2 + 10^2 - 2 \cdot 5 \cdot 10 \cos\theta = 125 + 100\cos\theta.$$

Solving for EF^2 we get $EF^2 = 200$.

8. [**1152**] Two parallel lines ℓ_1 and ℓ_2 are on a plane with distance d. On ℓ_1 there are infinitely many points $A_1, A_2, A_3, \cdots$ progressing in the same distance: $A_nA_{n+1} = 2$ for all n. In addition, on ℓ_2 there are also infinitely many points $B_1, B_2, B_3, \cdots$ satisfying $B_nB_{n+1} = 1$ for all n. Given that A_1B_1 is perpendicular to both ℓ_1 and ℓ_2, express the sum $\sum_{i=1}^{\infty} \angle A_iB_iA_{i+1}$ in terms of d.



Answer: $\boxed{\pi - \tan^{-1}\left(\frac{1}{d}\right)}$ Construct points $C_1, C_2, C_3, \ldots$ on ℓ_1 progressing in the same direction as the A_i such that $C_1 = A_1$ and $C_nC_{n+1} = 1$. Thus we have $C_1 = A_1$, $C_3 = A_2$, $C_5 = A_3$, etc., with $C_{2n-1} = A_n$ in general. We can write $\angle A_iB_iA_{i+1} = \angle C_{2i-1}B_iC_{2i+1} = \angle C_iB_iC_{2i+1} - \angle C_iB_iC_{2i-1}$. Observe that $\triangle C_iB_iC_k$ (for any k) is a right triangle with legs of length d and $k - i$, and $\angle C_iB_iC_k = \tan^{-1}\left(\frac{k-i}{d}\right)$. So $\angle C_iB_iC_{2i+1} - \angle C_iB_iC_{2i-1} = \tan^{-1}\left(\frac{i+1}{d}\right) - \tan^{-1}\left(\frac{i-1}{d}\right)$. The whole sum is therefore

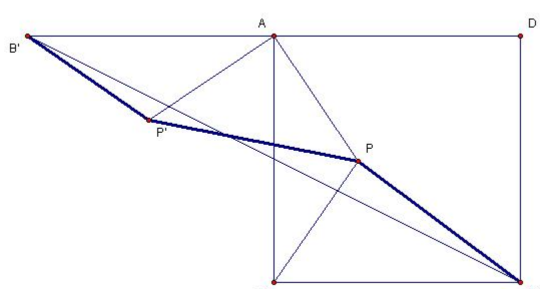
$$\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{i+1}{d}\right) - \tan^{-1}\left(\frac{i-1}{d}\right)$$

which has nth partial sum

$$\tan^{-1}\left(\frac{n+1}{d}\right) + \tan^{-1}\left(\frac{n}{d}\right) - \tan^{-1}\left(\frac{1}{d}\right)$$

so it converges to $\pi - \tan^{-1}\left(\frac{1}{d}\right)$.

9. [**1280**] In an unit square $ABCD$, find the minimum of $\sqrt{2}AP + BP + CP$ when P is an arbitrary point in $ABCD$.
Answer: $\boxed{\sqrt{5}}$ Rotate the triangle APB around A by 90 degree as in the following figure.

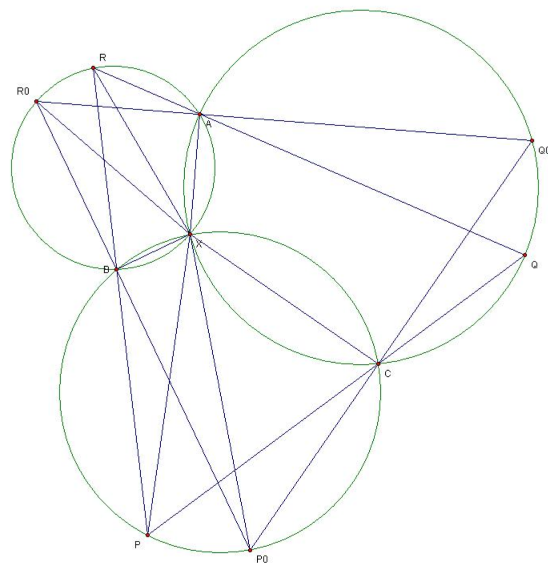


Let P' and B' be the rotated image of P and B respectively. Then we have $B'P' = BP$, $P'P = \sqrt{2}AP$ so

$$\sqrt{2}AP + BP + CP = CP = PP' + P'B' \leq CB' = \sqrt{5}.$$

10. [**1536**] Given a triangle ABC with side lengths $a = 5$, $b = 7$, $c = 8$, find the side length of largest equilateral triangle PQR such that A, B, C are on QR, RP, PQ, respectively.

Answer $\boxed{2\sqrt{43}}$ We claim that in general, the answer is $\sqrt{\frac{2}{3}(a^2+b^2+c^2+4\sqrt{3}S)}$, where S is the area of ABC. Suppose that PQR is a equilateral triangle satisfying the conditions. Then $\angle BPC = \angle CQA = \angle ARB = 60$. The locus of points satisfying $\angle BXC = 60$ is an part of a circle O_a. Draw O_b and O_c similarly. Three circles meet at a single point X inside the triangle, which is the unique point satisfying $\angle BXC = \angle CXA = \angle AXB = 120$. Then the choice of P on O_a determines Q and R: those two points should also be on O_b and O_c respectively, and PCQ and PBR should form the side of triangle. Now one should find the maximum of PQ under these conditions. Note that $\angle BPX$ and $\angle BRX$ does not depend on choice of P, so triangle PXR has same shape. Especially, the ratio of PX and PR is constant, so PR is maximized when PX is the diameter of O_a. This requires PQ, QR, RP to be perpendicular to XC, XA, XB respectively.



From this point there may be several ways to calculate the answer. One way is to observe $PQ = \frac{2}{\sqrt{3}}(AX + BX + CX)$ by considering $(PQR) = (PXQ) + (QXR) + (RXP)$. $AX + BX + CX$ can be computed by the usual rotation trick for the Fermat point: rotate $\triangle BXA$ $60°$ around B to $\triangle BX'A'$. Observe that $\triangle BXX'$ is equilateral, and so A', X', X, and C are collinear. Hence, $A'C = AX + BX + CX$, and we can apply the Law of Cosines to $\triangle A'BC$ to get that

$$A'C^2 = c^2 + a^2 - 2ac\cos(B + 60°) = a^2 + c^2 + 2ac\sin 60°\sin B - 2ac\cos 60°\cos B$$
$$= a^2 + c^2 + 2S\sqrt{3} - \frac{1}{2}(a^2 + c^2 - b^2) = \frac{a^2 + b^2 + c^2}{2} + 2S\sqrt{3}$$
$$\implies PQ = \sqrt{\frac{2}{3}(a^2 + b^2 + c^2 + 4\sqrt{3}S)}$$

(where S is again the area of ABC). Plugging in our values for a, b, and c, and using Heron's formula to find $S = \sqrt{10 \cdot 5 \cdot 3 \cdot 2} = 10\sqrt{3}$, we can calculate $PQ = 2\sqrt{43}$.