12^{th} Annual Johns Hopkins Math Tournament
Saturday, February 19, 2011

Geometry Subject Test

1. [**1025**] Let $D_{x,y}$ denote the half-disk of radius 1 with its curved boundary externally tangent to the unit circle at the point (x, y), such that the straight boundary of the disk is parallel to the tangent line (so the point of tangency is the middle of the curved boundary). Find the area of the union of the $D_{x,y}$ over all (x, y) with $x^2 + y^2 = 1$ (that is, (x, y) is on the unit circle).

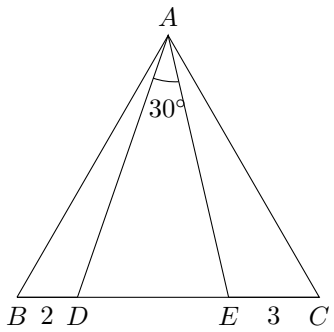
2. [**1026**] Let circle O have radius 5 with diameter AE. Point F is outside circle O such that lines AF and EF intersect circle O at points B, and D, respectively. If $AF = 10$ and $m\angle FAE = 30°$, then the perimeter of quadrilateral $ABDE$ can be expressed as $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, where a, b, c, d are rational. Find $a + b + c + d$.

3. [**1028**] In a unit cube $ABCD - EFGH$, an equilateral triangle BDG cuts out a circle from the circumsphere of the cube. Find the area of the circle.

4. [**1032**] Compute the largest value of r such that three non-overlapping circles of radius r can be inscribed in a unit square.

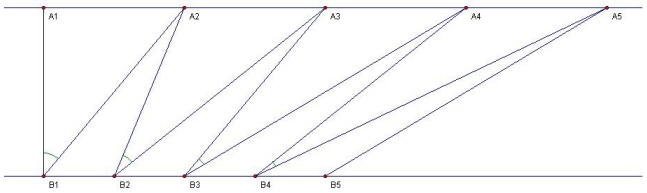
5. [**1040**] Let $ABCD$ be a unit square. Point E is on BC, point F is on DC, $\triangle AEF$ is equilateral, and $GHIJ$ is a square in $\triangle AEF$ such that GH is on EF. Compute the area of square $GHIJ$.

6. [**1056**] Let $\triangle ABC$ be equilateral. Two points D and E are on side BC (with order B, D, E, C), and satisfy $\angle DAE = 30°$. If $BD = 2$ and $CE = 3$, what is BC?



7. [**1088**] Let $ABCD$ be a cyclic quadrilateral with $AB = 6$, $BC = 12$, $CD = 3$, and $DA = 6$. Let E, F be the intersection of lines AB and CD, lines AD and BC, respectively. Find EF.

8. [**1152**] Two parallel lines ℓ_1 and ℓ_2 are on a plane with distance d. On ℓ_1 there are infinitely many points $A_1, A_2, A_3, \cdots$ progressing in the same distance: $A_nA_{n+1} = 2$ for all n. In addition, on ℓ_2 there are also infinitely many points $B_1, B_2, B_3, \cdots$ satisfying $B_nB_{n+1} = 1$ for all n. Given that A_1B_1 is perpendicular to both ℓ_1 and ℓ_2, express the sum $\sum_{i=1}^{\infty} \angle A_iB_iA_{i+1}$ in terms of d.



9. [**1280**] In an unit square $ABCD$, find the minimum of $\sqrt{2}AP + BP + CP$ when P is an arbitrary point in $ABCD$.

10. [**1536**] Given a triangle ABC with side lengths $a = 5$, $b = 7$, $c = 8$, find the side length of largest equilateral triangle PQR such that A, B, C are on QR, RP, PQ, respectively.