12^{th} Annual Johns Hopkins Math Tournament

Saturday, February 19, 2011

General Test 2

1. [**1025**] Let $a, b \in \mathbb{C}$ such that $a + b = a^2 + b^2 = \frac{2\sqrt{3}}{3}i$. Compute $|\text{Re}(a)|$.

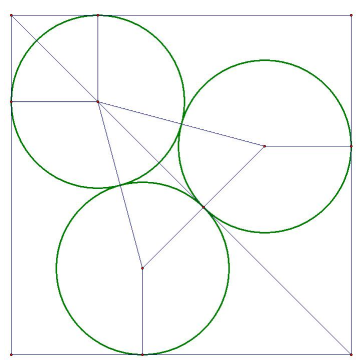
   Answer: $\boxed{\dfrac{1}{\sqrt{2}}}$ From $a + b = \frac{2\sqrt{3}}{3}i$ we can let $a = x + \frac{\sqrt{3}}{3}i$ and $b = -x + \frac{\sqrt{3}}{3}i$. Then $a^2 + b^2 = 2(i^2 + x^2) = 2\left(x^2 - \frac{1}{3}\right) = \frac{2\sqrt{3}}{3}i$. So $x^2 = \frac{1+\sqrt{3}i}{3} = \frac{2}{3}e^{i\pi/3}$, and $x = \pm\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}+i}{2}$. Since $|\text{Re}(a)| = |\text{Re}(x)|$, the answer is $\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}}$.

2. [**1026**] You are given a dart board with a small circle that is worth 20 points and a ring surrounding the circle that is worth 11 points. No points are given if you do not hit any of these areas. What is the largest integeral number of points that cannot be achieved with some combination of hits.

   Answer: $\boxed{189}$ Because 20 and 11 are relatively prime, the largest number than cannot be expressed as $am + bn$ for positive integers a and b is $mn - m - n$, so the answer is $20 \cdot 11 - 20 - 11 = 189$.

3. [**1028**] Compute the largest value of r such that three non-overlapping circles of radius r can be inscribed in a unit square.

   Answer: $\boxed{\dfrac{\sqrt{2}}{1 + 2\sqrt{2} + \sqrt{3}}}$ The three circles will be inscribed in such a way that one altitude of the equilateral triangle formed by the centers of the three circles will coincide with a diagonal of the square, as in the figure below. Indeed, by the Pigeonhole Principle, one circle must lie tangent to two sides of the square, and in any orientation other than the one below, the circles can be dilated.



   Label the square $ABCD$ starting in the upper left and going clockwise. The line from the center of the top left circle to A has length $\sqrt{2}r$, and the equilateral triangle formed by the radii has height $\sqrt{3}r$. Then the line from the base of the equilateral triangle to C has length $\sqrt{2} - (\sqrt{2} + \sqrt{3})r$. Now, draw lines from the centers of the two lower circles to C to form four triangles. Observe that these triangles are identical, with angle $\frac{\pi}{8}$ in the lower right. Then

$$\tan\left(\frac{\pi}{8}\right) = \sqrt{2} - 1 = \frac{r}{\sqrt{2} - (\sqrt{2} + \sqrt{3})r}$$

   and solving for r gives the desired answer, or some equivalent expression.

4. [**1032**] Find the ten smallest x, with $x > 1$, that satisfy the following relation:

$$\sin(\ln x) + 2\cos(3\ln x)\sin(2\ln x) = 0$$

   Answer: $\boxed{x = e^{n\pi/5} \text{ for } n = 1, 2, \ldots, 10}$ Set $y = \ln x$, and observe that

$$2\cos(3y)\sin(2y) = \sin(3y + 2y) - \sin(3y - 2y) = \sin(5y) - \sin(y),$$

so that the equation in question is simply

$$\sin(5y) = 0.$$

The solutions are therefore

$$\ln x = y = \frac{n\pi}{5} \implies x = e^{n\pi/5} \quad \text{for } n = 1, 2, \ldots, 10.$$

5. [**1040**] If r, s, t, and u denote the roots of the polynomial $f(x) = x^4 + 3x^3 + 3x + 2$, find

$$\frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2} + \frac{1}{u^2}$$

Answer: $\boxed{\frac{9}{4}}$ First notice that the polynomial

$$g(x) = x^4\left(\frac{1}{x^4} + \frac{3}{x^3} + \frac{3}{x} + 2\right) = 2x^4 + 3x^3 + 3x + 1$$

is a polynomial with roots $\frac{1}{r}$, $\frac{1}{s}$, $\frac{1}{t}$, $\frac{1}{u}$. Therefore, it is sufficient to find the sum of the squares of the roots of $g(x)$. Let s_1 denote the sum of the roots of $g(x)$, and let s_2 equal the sum of the squares of the roots of $g(x)$. Since $x^2 + y^2 = (x+y)^2 - 2xy$, we have that

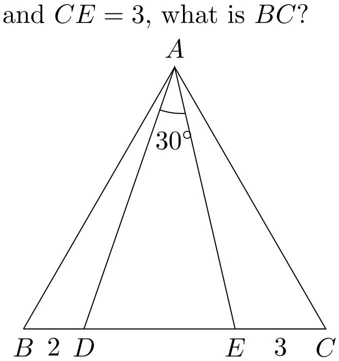
$$a_4 s_1 + a_3 = 0 \text{ and } a_4 s_2 + a_3 s_1 + 2a_2 = 0,$$

where a_n denotes the coefficient of x^n in a polynomial. Therefore, applying this to $g(x)$, we have that

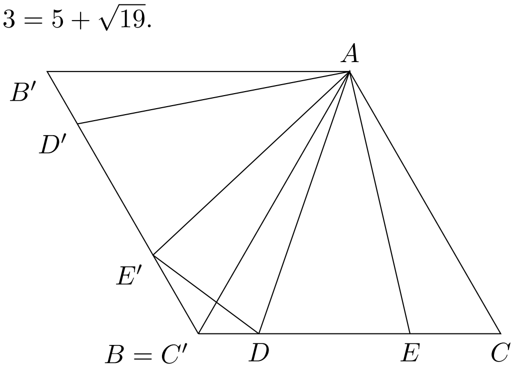
$$2s_1 + 3 = 0 \implies s_1 = -\frac{3}{2}$$

$$2s_2 + 3\left(-\frac{3}{2}\right) + 2(0) = 2s_2 - \frac{9}{2} = 0 \implies s_2 = \frac{9}{4}.$$

6. [**1056**] Let $\triangle ABC$ be equilateral. Two points D and E are on side BC (with order B, D, E, C), and satisfy $\angle DAE = 30°$. If $BD = 2$ and $CE = 3$, what is BC?

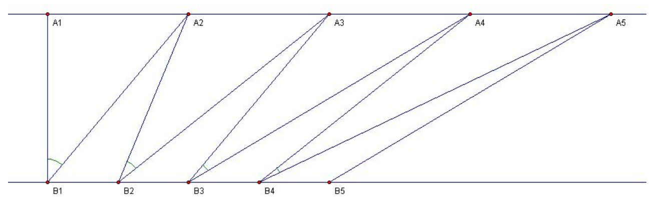


Answer: $\boxed{5 + \sqrt{19}}$ Rotate the figure around A by $60°$ so that C comes at the place of B. Let B', C', D', E' be corresponding points of the moved figure. Since $\angle E'AD = \angle E'AC' + \angle C'AD = \angle EAC + \angle BAD = 30° = \angle EAD$, $E'A = EA$ and $DA = D'A$, one has $E'D = ED$. So $BC = BD + DE + EA$ can be found out if we know $E'D$. But $E'D = \sqrt{E'B^2 + BD^2 - 2 \cdot E'B \cdot BD \cdot \cos 120°} = \sqrt{19}$, so $BC = 2 + \sqrt{19} + 3 = 5 + \sqrt{19}$.



7. [**1088**] Two ants, Yuri and Jiawang, begin on opposite corners of a cube. On each move, they can travel along an edge to an adjacent vertex. Find the probability they both return to their starting position after 4 moves.
Answer: $\boxed{\dfrac{49}{729}}$ Let the cube be oriented so that one ant starts at the origin and the other at $(1,1,1)$. Let x, y, z be moves away from the origin and x', y', z' be moves toward the origin in each the respective directions. Any move away from the origin has to at some point be followed by a move back to the origin, and if the ant moves in all three directions, then it can't get back to its original corner in 4 moves. The number of ways to choose 2 directions is $\binom{3}{2} = 3$ and for each pair of directions there are $\frac{4!}{2!2!} = 6$ ways to arrange four moves a, a', b, b' such that a precedes a' and b precedes b'. Hence there are $3 \cdot 6 = 18$ ways to move in two directions. The ant can also move in a, a', a, a' (in other words, make a move, return, repeat the move, return again) in three directions so this gives $18 + 3 = 21$ moves. There are $3^4 = 81$ possible moves, 21 of which return the ant for a probability of $\frac{21}{81} = \frac{7}{27}$. Since this must happen simultaneously to both ants, the probability is $\frac{7}{27} \cdot \frac{7}{27} = \frac{49}{729}$.

8. [**1152**] Two parallel lines ℓ_1 and ℓ_2 are on a plane with distance d. On ℓ_1 there are infinitely many points $A_1, A_2, A_3, \cdots$ progressing in the same distance: $A_nA_{n+1} = 2$ for all n. In addition, on ℓ_2 there are also infinite points $B_1, B_2, B_3, \cdots$ satisfying $B_nB_{n+1} = 1$ for all n. Given that A_1B_1 is perpendicular to both ℓ_1 and l_2, express the sum $\sum_{i=1}^{\infty} \angle A_iB_iA_{i+1}$ in terms of d.



Answer: $\boxed{\pi - \tan^{-1}\left(\dfrac{1}{d}\right)}$ Construct points $C_1, C_2, C_3, \ldots$ on ℓ_1 progressing in the same direction as the A_i such that $C_1 = A_1$ and $C_nC_{n+1} = 1$. Thus we have $C_1 = A_1$, $C_3 = A_2$, $C_5 = A_3$, etc., with $C_{2n-1} = A_n$ in general. We can write $\angle A_iB_iA_{i+1} = \angle C_{2i-1}B_iC_{2i+1} = \angle C_iB_iC_{2i+1} - \angle C_iB_iC_{2i-1}$. Observe that $\triangle C_iB_iC_k$ (for any k) is a right triangle with legs of length d and $k - i$, and $\angle C_iB_iC_k = \tan^{-1}\left(\frac{k-i}{d}\right)$. So $\angle C_iB_iC_{2i+1} - \angle C_iB_iC_{2i-1} = \tan^{-1}\left(\frac{i+1}{d}\right) - \tan^{-1}\left(\frac{i-1}{d}\right)$. The whole sum is therefore

$$\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{i+1}{d}\right) - \tan^{-1}\left(\frac{i-1}{d}\right)$$

which has nth partial sum

$$\tan^{-1}\left(\frac{n+1}{d}\right) + \tan^{-1}\left(\frac{n}{d}\right) - \tan^{-1}\left(\frac{1}{d}\right)$$

so it converges to $\pi - \tan^{-1}\left(\frac{1}{d}\right)$.

9. [**1280**] Let $\{a_i\}_{i=1,2,3,4}, \{b_i\}_{i=1,2,3,4}, \{c_i\}_{i=1,2,3,4}$ be permutations of $\{1, 2, 3, 4\}$. Find the minimum of $a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3 + a_4b_4c_4$.
Answer: $\boxed{44}$ The minimum can be obtained by

$$1 \cdot 3 \cdot 4 + 2 \cdot 2 \cdot 3 + 3 \cdot 4 \cdot 1 + 4 \cdot 1 \cdot 2 = 12 + 12 + 12 + 8 = 44.$$

We claim that 44 is optimum. Denote $x_i = a_ib_ic_i$. Since $x_1x_2x_3x_4 = (1 \cdot 2 \cdot 3 \cdot 4)^3 = 2^9 \cdot 3^3$, x_i should only consist of prime factors of 2 and 3. So between 8 and 12 x_i can only be 9.

(a) There are no 9 among x_i. Then x_i are not in $(8, 12)$. And $x_1x_2x_3x_4 = 12 \cdot 12 \cdot 12 \cdot 8$, so if x_1 is minimum then $x_1 \leq 8$. Then by the AM-GM inequality $x_2 + x_3 + x_4 \geq 3(x_2x_3x_4)^{1/3}$. If we let $(x_2x_3x_4)^{1/3} = 12y$, then $x_1 = 8y^{-3}$, and for $y \geq 1$, $8y^{-3} + 36y$ attains its minimum at $y = 1$. So $x_1 + x_2 + x_3 + x_4 \geq 8y^{-3} + 36y \geq 44$.

(b) x_1 is 9. Then $x_2x_3x_4$ is divisible by 3 but not 9. So only x_2 is divisible by 3 and others are just powers of 2. x_2 can be 3, 6, 12, 24 or larger than 44. Consider the following subcases:

i. $x_2 = 3$: $x_3x_4 = 2^9$, $x_3 + x_4 \leq 2^5 + 2^4 = 48 > 44$.

ii. $x_2 = 6$: $x_3x_4 = 2^8$, $x_3 + x_4 \leq 2^4 + 2^4 = 32$, $x_1 + x_2 + x_3 + x_4 \leq 9 + 6 + 32 = 47$.

iii. $x_2 = 12$: $x_3x_4 = 2^7$, $x_3 + x_4 \leq 2^4 + 2^3 = 24$, $x_1 + x_2 + x_3 + x_4 \leq 9 + 12 + 24 = 45$.

iv. $x_2 = 24$: $x_3x_4 = 2^6$, $x_3 + x_4 \leq 2^3 + 2^3 = 16$, $x_1 + x_2 + x_3 + x_4 \leq 9 + 24 + 16 = 49$.

10. [**1536**] How many functions f that take $\{1, 2, 3, 4, 5\}$ to itself, i.e. that is permutes the set, satisfy $f(f(f(x))) = f(f(x))$ for all x in $\{1, 2, 3, 4, 5\}$?

Answer: $\boxed{756}$ For any such function f, let $A = \{n \mid f(n) = n\}$ be the set of elements fixed by f and let $B = \{n \mid f(n) \in A \text{ and } n \notin A\}$ be the set of elements that are sent to an element in A, but are not themselves in A. Finally, let $C = \{1, 2, 3, 4, 5\} \setminus (A \cup B)$ be everything else. Note that any possible value of $f(f(x))$ is in A so A is not empty. We will now proceed by considering all possible sizes of A.

(a) A has one element: Without loss of generality, let $f(1) = 1$, so we will mutliply our result by 5 at the end to account for the other possible values. Suppose that B has n elements so C has the remaining $4 - n$ elements. Since $f(f(x)) = 1$ for each x so any element c in C must satisfy $f(c) = b$ for some b in B, because $f(c) \neq 1$ and the only other numbers for which $f(x) = 1$ are the elements of B. This also implies that B is not empty. Conversely, any function satisfying $f(c) = b$ works, so the total number of functions in this case is $5 \sum_{n=1}^{4} \binom{4}{n} n^{4-n}$ because there are $\binom{4}{n}$ ways to choose the elements in B, and each of the $4 - n$ elements in C can be sent to any element of B (there are n of them). This sum is equal to $5(4 + 6 \cdot 4 + 4 \cdot 3 + 1) = 205$, so there are 205 functions in this case that A has one element.

(b) A has two elements: This is similar to the first case, except that each element in B can now correspond to one of two possible elements in A, so this adds a factor of 2^n. The sum now becomes $\binom{5}{2} \sum_{n=1}^{3} \binom{3}{n} 2^n n^{3-n} = 10(3 \cdot 2 + 3 \cdot 4 \cdot 2 + 8) = 380$, so there are 380 functions in this case.

(c) A has three elements: This is again similar to the prior cases, except there are 3 possible targes in A, adding a factor of 3^n. Then the sum is $\binom{5}{3} \sum_{n=1}^{2} \binom{2}{n} 3^n n^{2-n} = 10(2 \cdot 3 + 9) = 150$, so there are 150 functions in this case.

(d) A has four elements: The logic is the same as the prior cases and there are $5(4) = 20$ functions in this case.

(e) A has five elements: The identity function is the only possible function in this case.

Adding together the five cases, we see that there are $205 + 380 + 150 + 20 + 1 = 756$ such functions.