12^{th} Annual Johns Hopkins Math Tournament
Saturday, February 19, 2011

General Test 1

1. [**1025**] Let $F(x)$ be a real-valued function defined for all real $x \neq 0, 1$ such that

$$F(x) + F\left(\frac{x-1}{x}\right) = 1 + x.$$

Find $F(2)$.

Answer: $\boxed{\frac{3}{4}}$ Setting $x = 2$, we find that $F(2) + F\left(\frac{1}{2}\right) = 3$. Now take $x = \frac{1}{2}$, to get that $F\left(\frac{1}{2}\right) + F(-1) = \frac{3}{2}$. Finally, setting $x = -1$, we get that $F(-1) + F(2) = 0$. Then we find that

$$\begin{aligned} F(2) &= 3 - F\left(\frac{1}{2}\right) = 3 - \left(\frac{3}{2} - F(-1)\right) = \frac{3}{2} + F(-1) = \frac{3}{2} - F(2) \\ &\Rightarrow F(2) = \frac{3}{4}. \end{aligned}$$

Alternate Solution: We can explicitly solve for $F(x)$ and then plug in $x = 2$. Notice that for $x \neq 0, 1$, $F(x) + F\left(\frac{x-1}{x}\right) = 1 + x$ so

$$F\left(\frac{x-1}{x}\right) + F\left(\frac{1}{1-x}\right) = 1 + \frac{x-1}{x} \text{ and } F\left(\frac{1}{1-x}\right) + F(x) = 1 + \frac{1}{1-x}.$$

Thus

$$\begin{aligned} 2F(x) &= F(x) + F\left(\frac{x-1}{x}\right) - F\left(\frac{x-1}{x}\right) - F\left(\frac{1}{1-x}\right) + F\left(\frac{1}{1-x}\right) + F(x) \\ &= 1 + x - \left(1 + \frac{x-1}{x}\right) + 1 + \frac{1}{1-x} \\ &= 1 + x + \frac{1-x}{x} + \frac{1}{1-x}. \end{aligned}$$

It follows that $F(x) = \frac{1}{2}\left(1 + x + \frac{1-x}{x} + \frac{1}{1-x}\right)$ and the result follows by taking $x = 2$.

2. [**1026**] Find the number of pairs (a, b) with a, b positive integers such that $\frac{a}{b}$ is in lowest terms and $a + b \leq 10$.

Answer: $\boxed{32}$ If $\frac{a}{b}$ is in lowest terms, then it means that a and b are relatively prime; i.e. their greatest common divisor is 1. Now, a and b are relatively prime if and only if a and $a + b$ are relatively prime (indeed, if a number divides a and $a + b$ then it must also divide b). So now we just need to count the number of integers relatively prime to each of $1, 2, \ldots, 10$, which is given by

$$\sum_{i=1}^{10} \varphi(i) = 1 + 1 + 2 + 2 + 4 + 2 + 6 + 4 + 6 + 4 = 32,$$

where $\varphi(n)$ is the Euler-phi function.

3. [**1028**] Find all rational roots of $|x - 1|\left|x^2 - 2\right| - 2 = 0$.

Answer: $\boxed{-1, 0, 2}$ There are four intervals to consider, each with their own restrictions. Consider the case in which $x > \sqrt{2}$. Then the equation becomes $(x-1)(x^2-2) - 2 = x(x-2)(x+1) = 0$. Thus, $x = 2$ is the only rational root for $x > \sqrt{2}$. Consider the case in which $-\sqrt{2} < x < 1$. Then the equation becomes $(x-1)(x^2-2) - 2 = x(x-2)(x+1) = 0$. Thus, $x = 0$ and $x = -1$ are the rational roots for $-\sqrt{2} < x < 1$. Consider the case in which $x < -\sqrt{2}$ or the case in which $1 < x < \sqrt{2}$. In these cases, the equation becomes $(1-x)(x^2-2) - 2 = -x^3 + x^2 + 2x - 4$. By the rational root theorem, the rational roots of this polynomial can only be $\pm 4, \pm 2, \pm 1$ and a quick check shows that none of these are roots, so this polynomial has no rational roots.

4. [**1032**] Let $M = (-1, 2)$ and $N = (1, 4)$ be two points in the plane, and let P be a point moving along the x-axis. When $\angle MPN$ takes on its maximum value, what is the x-coordinate of P?
Answer: $\boxed{1}$ Let $P = (a, 0)$. Note that $\angle MPN$ is inscribed in the circle defined by points M, P, and N, and that it intercepts MN. Since MN is fixed, it follows that maximizing the measure of $\angle MPN$ is equivalent to minimizing the size of the circle defined by M, P, and N. Since P must be on the x-axis, we therefore want this circle to be tangent to the x-axis. Since the center of this circle must lie on the perpendicular bisector of MN, which is the line $y = 3 - x$, the center of the circle has to be of the form $(a, 3 - a)$, so a has to satisfy $(a + 1)^2 + (1 - a)^2 = (a - 3)^2$. Solving this equation gives $a = 1$ or $a = -7$. Clearly choosing $a = 1$ gives a smaller circle, so our answer is 1.

5. [**1040**] Mordecai is standing in front of a 100-story building with two identical glass orbs. He wishes to know the highest floor from which he can drop an orb without it breaking. What is the minimum number of drops Mordecai can make such that he knows for certain which floor is the highest possible?
Answer: $\boxed{14}$ Consider dropping the orb from the nth floor. If the orb breaks, then we should go down to the lowest floor from which we know it will not break. In this case, that would be ground level so go to the first floor and drop the second orb. If it breaks, we are done. Otherwise, we go up to the second floor and continue. In this case, it will take no more than n drops to find the desired floor. Now, suppose that the orb did not break when dropped from the nth floor. Go up to floor $n + k$. If the orb breaks, go to floor $n+1$ (because we know it won't break on floor n). If it breaks, we're done; otherwise, proceed to $n + 2$ and proceed as before. In this case, it will take at most $2 + k - 1 = k + 1$ drops. But this should not require any more drops than the first time, so we have $n = k + 1$, or $k = n - 1$. Now, if the orb did not drop on the $n + k$th floor, proceed up to the $n + k + \ell = 2n + \ell - 1$th floor. Repeat the process. We can conclude that $3 + \ell - 1 = n$, or $\ell = n - 2$. Continuing inductively, we will ultimately end up on floor $n + (n - 1) + (n - 2) + \cdots + 1 = \frac{n(n+1)}{2}$ (assuming the orb never broke). The desired n is the smallest one such that $\frac{n(n+1)}{2} > 100$, because there are 100 floors. This is easily computed to be $n = 14$.

6. [**1056**] Let ABC be any triangle, and D, E, F be points on BC, CA, AB such that $CD = 2BD$, $AE = 2CE$ and $BF = 2AF$. Also, AD and BE intersect at X, BE and CF intersect at Y, and CF and AD intersect at Z. Find the ratio of the areas of $\triangle ABC$ and $\triangle XYZ$.
Answer: $\boxed{7}$ Using Menelaus' Theorem on $\triangle ABD$ with collinear points F, X, C and the provided ratios gives $\frac{DX}{XA} = \frac{4}{3}$. Using Menelaus' Theorem on $\triangle ADC$ with collinear points B, Y, E gives $\frac{AY}{YD} = 6$. We conclude that AX, XY, YD are in length ratio $3 : 3 : 1$. By symmetry, this also applies to the segments CZ, ZX, XF and BY, YZ, ZE. Repeatedly using the fact that the area ratio of two triangles of equal height is the ratio of their bases, we find $[ABC] = \frac{3}{2}[ADC] = \frac{3}{2}\frac{7}{3}[XYC] = \frac{3}{2}\frac{7}{3} \cdot 2[XYZ] = 7[XYZ]$, or $\frac{[ABC]}{[XYZ]} = 7$.
Alternate Solution: Stretching the triangle will preserve ratios between lengths and ratios between areas, so we may assume that $\triangle ABC$ is equilateral with side length 3. We now use mass points to find the length of XY. Assign a mass of 1 to A. In order to have X be the fulcrum of $\triangle ABC$, we must ahve C have mass 2 and B must have mass 4. Hence, $BX : XE = 4 : 3$ and $AX : XD = 6 : 1$, the latter of which also equals $BY : YE$ by symmetry. Hance, $XY = \frac{3}{7}BE$. To find BE, we apply the Law of Cosines to $\triangle CBE$ to get that

$$BE^2 = 1^1 = 3^2 - 2 \cdot 1 \cdot 3 \cos 60 = 7 \Rightarrow XY = \frac{3\sqrt{7}}{7}.$$

Since $\triangle XYZ$ must be equilateral by symmetry, the desired ratio is $\left(\frac{AB}{XY}\right)^2 = 7$.

7. [**1088**] Find the projection of the sphere $x^2 + y^2 + (z - 1)^2 = 1$ onto the plane $z = 0$ with respect to the point $P = (0, -1, 2)$. [The answer will be some conic curve. Express the equation in the form $f(x, y) = 0$ where f is some quadratic in x and y.]
Answer: $\boxed{x^2 - 4y - 4 = 0}$ Let $O = (0, 0, 1)$ be the center of the sphere. For a point $X = (x, y, 0)$ on the boundary of the projection, the angle $\angle XPO$ is constant as X varies, since it is just the angle between OP and any tangent from P to the sphere. Considering the case when $X = (0, -1, 0)$, we can see that $\angle XPO = 45°$. Writing this in terms of the dot product, one has $(\overrightarrow{PO} \cdot \overrightarrow{PX})^2 = \frac{1}{2}|\overrightarrow{PO}|^2|\overrightarrow{PX}|^2$, which is equivalent to $((0, 1, -1) \cdot (x, y + 1, -2))^2 = \frac{1}{2}|(0, 1, -1)|^2|(x, y + 1, -2)|^2$, or $(y + 3)^2 = x^2 + (y + 1)^2 + 4$. The answer is $x^2 - 4y - 4 = 0$.

8. [**1152**] It is a well-known fact that the sum of the first n kth powers can be represented as polynomials in n. Let $P_k(n)$ be such polynomial. For example, one has $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, so one has $P_2(x) = \frac{x(x+1)(2x+1)}{6} = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x$. Evaluate $P_3(-4) + P_4(-3)$.
Answer: $\boxed{19}$ Since the equation

$$P_k(x) = P_k(x-1) + x^k$$

has all integers ≥ 2 as roots, it should be identity, so it holds for all x. Now we can substitute $x = -1, -2, -3, -4, \ldots$ to prove

$$P_k(-n) = \sum_{i=1}^{n-1} -(-i)^k.$$

Therefore, $P_3(-4) + P_4(-3) = -(-1)^3 - (-2)^3 - (-3)^3 - (-1)^4 - (-2)^4 = 19$.

9. [**1280**] Find the final non-zero digit in 100!. For example, the final non-zero digit of 7200 is 2.
Answer: $\boxed{4}$ We first claim that 100! ends in 24 zeroes. Indeed, it suffices to count the number of 5's in the prime factorization of 100!. There are 20 multiples of 5 up to 100, which gives 20 zeroes, and then 25, 50, 75, and 100 each contribute one more for a total of 24. Now, let $p(k)$ denote the product of the first k positive multiples of 5, and notice that $p(k) = 5^k \cdot k!$. Also, by cancelling terms of $p(k)$, we have that $\frac{(5k)!}{p(k)} \equiv (1 \cdot 2 \cdot 3 \cdot 4)^k \equiv (-1)^k \pmod 5$. From our claim, we can write $100! = M \cdot 10^{24}$, where

$$M = 2^{-24} \cdot \frac{100!}{p(20)} \cdot \frac{20!}{p}(4) \cdot 4! \equiv (2^4)^{-6} \cdot (-1)^{20} \cdot (-1)^4 \cdot (-1) \equiv -1 \pmod 5.$$

Since more 2's than 5's divide 100!, the last nonzero digit must be even, and so it is 4.

10. [**1536**] How many polynomials P of degree 4 satisfy $P(x^2) = P(x)P(-x)$?
Answer: $\boxed{10}$ Note that if r is a root of P then r^2 is also a root. Therefore $r, r^2, r^{2^2}, r^{2^3}, \cdots$, are all roots of P. Since P has a finite number of roots, two of these roots should be equal. Therefore, either $r = 0$ or $r^N = 1$ for some $N > 0$. If all roots are equal to 0 or 1, then P is of the form $ax^b(x-1)^{(4-b)}$ for $b = 0, ..., 4$. Now suppose this is not the case. For such a polynomial, let q denote the largest integer such that $r = e^{2\pi i \cdot p/q}$ is a root for some integer p coprime to q. We claim that the only suitable $q > 1$ are $q = 3$ and $q = 5$. First note that if r is a root then one of $\sqrt{r}$ or $-\sqrt{r}$ is also a root. So if q is even, then one of $e^{2\pi i \cdot p/2q}$ or $e^{2\pi i \cdot p+q/2q}$ should also be root of p, and both p/q and $(p+q)/2q$ are irreducible fractions. This contradicts the assumption that q is maximal. Therefore q must be odd. Now, if $q > 6$, then $r^{-2}, r^{-1}, r, r^2, r^4$ should be all distinct, so $q \leq 6$. Therefore $q = 5$ or 3. If $q = 5$, then the value of p is not important as P has the complex fifth roots of unity as its roots, so $P = a(x^4 + x^3 + x^2 + x + 1)$. If $q = 3$, then P is divisible by $x^2 + x + 1$. In this case we let $P(x) = a(x^2 + x + 1)Q(x)$ and repeating the same reasoning we can show that $Q(x) = x^2 + x + 1$ or $Q(x)$ is of form $x^b(x-1)^{2-b}$. Finally, we can show that exactly one member of all 10 resulting families of polynomials fits the desired criteria. Let $P(x) = a(x-r)(x-s)(x-t)(x-u)$. Then, $P(x)P(-x) = a^2(x^2 - r^2)(x^2 - s^2)(x^2 - t^2)(x^2 - u^2)$. We now claim that r^2, s^2, t^2, and u^2 equal r, s, t, and u in some order. We can prove this noting that the mapping $f(x) = x^2$ maps 0 and 1 to themselves and maps the third and fifth roots of unity to another distinct third or fifth root of unity, respectively. Hence, for these polynomials, $P(x)P(-x) = a^2(x^2 - r)(x^2 - s)(x^2 - t)(x^2 - u) = aP(x^2)$, so there exist exactly 10 polynomials that fit the desired criteria, namely the ones from the above 10 families with $a = 1$.