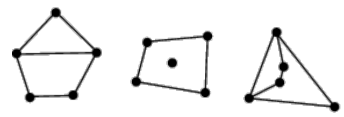
12^{th} Annual Johns Hopkins Math Tournament
Saturday, February 19, 2011
Explorations Unlimited Round-The DNA Inequality

Problem 1.(20=5+15)

(1) 5



the three cases for 5 points

(2) We construct a set of 2^{n-2} points in the plane without a convex n-gon. For all sets, no three points are collinear. We call the sequence

$$(x_i, y_i)\ i = 0, ...k \text{ and } x_0 < x_1 < ... < x_k$$

convex of length k if

$$\frac{y_i - y_{i-1}}{x_i - x_{i-1}} < \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \text{ for } i = 1, ..., k-1$$

Suppose P_i $(i = 1, ..., r)$ is a non-empty subset of S_{k_i} $1 \le k_1 < ...k_r \le n-1$ such that $P = \cup_{i=1}^r P_i$ forms a convex polygon. Since the slope of lines within each S_k is positive, P_i for $1 < i < r$ consists of a single point and P_1 must form a concave sequence P_r a convex sequence. Then the total number of points in P is at most $k_1 + (k_r - k_1 - 1) + (n - k_r) = n - 1$.

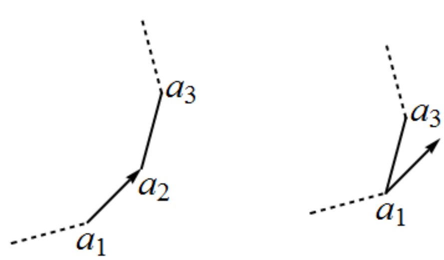
Problem 2.(12=4+4+4)

(1) $n = 2$. If $n < 2$ then clearly there can be disjoint sets so the entire intersection will be empty. If $n = 2$ any two intervals will have a non-empty intersection so it follows that if $a < b < c < d$ and $(a, c) \cap (b, d)$ is not empty, that any set (e, f) must have non empty intersection with both other intervals. Hence either we must have $e < c < d$ or $f > b$ so it must be the case that $(a, c) \cap (b, d) \cap (e, f)$ is not empty. Hence $n = 2$ is sufficient.

(2) $n = 3$, A similar argument applies for the remaining parts but one must consider one more degree of freedom in each case.

(3) $n = 4$

Problem 3.(10)



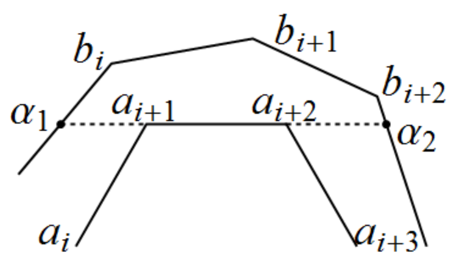
(1) Take a polygonal curve γ with n line segments and vertices $(a_1, a_2, ..., a_n)$. Fix a_1 and slide the entire curve with $a_2 \to a_1$ such that the angle between $\overline{a_1a_2}$ and $\overline{a_2a_3}$ is preserved. Continue this operation until all $a_i \to a_1$. Because angles were preserved, the total curvature of γ is equal to the sum of angles between the segments $\overline{a_ia_{i+1}}$ and $\overline{a_{i+1}a_{i+2}}$. Because the angles are between each pair of segments they must begin and end around $\overline{a_1a_2}$. It can easily be seen that this total angle is a multiple of a full turn of the unit circle.

Problem 4.(**12=6+6**) The left figure has curvature 2π, and the right curvature 4π. Because they have length 1 and have no negative curvature, these are also their average absolute curvatures.

Problem 5.(**18=2+2+4+4+3+3**)

(1) The average absolute curvature of γ is $A(\gamma) = \frac{K(\gamma)}{L(\gamma)}$. Thus, the average absolute curvature of Γ is $A(\Gamma) = \frac{2\pi}{2\pi(1)} = 1$.
(2) Theorem 4.1 gives us that $A(\gamma) \geq A(\Gamma)$. Therefore $\frac{K(\gamma)}{L(\gamma)} \geq 1$. Hence $K(\gamma) \geq L(\gamma)$.
(3) When unfolded the distance is simply $L(\gamma)$ because the entire curve is straightened to a line while preserving length.
(4) Largest radius is 1 because the curve is contained inside the unit circle. Thus, the length is ϕ_i.
(5) The length is the length of all the arcs $\sum \phi_i(r_i)$.
(6) We have $L(\gamma) = \sum \phi_i r_i \leq \sum \phi_i = K(\gamma)$.

Problem 6.(**20**) Let A have vertices $\{a_1, ..., a_n\}$ and B $\{b_1, ..., b_n\}$ we do the following procedure: extend $\overline{a_{i+1}a_{i+2}}$ until it intersects B. Now modify B by moving the part enclosed between the intersection points of A's extension to the extension. That is $(\alpha_1, b_i, b_{i+2}, \alpha_2) \to (\alpha_1, \alpha_2)$. Now extend $\overline{a_{i+2}a_{i+3}}$ in a similar manner .... repeat until $B \to A$. Now, throughout this process $B \to B^1 \to ... \to A$. B^i are convex curves implies that they all have total absolute curvature 2π. If $A(\gamma)$ is average absolute curvature of γ, we need $A(B) \leq A(B^1) \leq ... \leq A(A)$. But because all curvatures are 2π, this means $L(B) \geq L(B^1) \geq ... \geq L(A)$. This follows because each modification is a reduction in length by the Δ-inequality.



Problem 7.(**18=2×9**)

(1)

$$\frac{2\pi}{\overline{AB} + \overline{BC} + \overline{CD} + \overline{DA}}$$

(2)

$$\frac{(\pi - \beta) + (\pi - \alpha) + (\pi - \delta) + (\pi - \gamma)}{\overline{AB} + \overline{BC} + \overline{CD} + \overline{DA}} = \frac{2\pi + 2\phi}{\overline{AB} + \overline{BC} + \overline{CD} + \overline{DA}}$$

(3) By the law of sines $\frac{\sin(\alpha)}{BC} = \frac{\sin(\gamma)}{AB} = \frac{\sin(\beta)}{AC}$ so

$$\frac{AB + BC}{AC} = \frac{AC\frac{\sin(\gamma)}{\sin(B)} + AC\frac{\sin(\alpha)}{\sin(B)}}{AC} = \frac{\sin(\gamma) + \sin(\alpha)}{\sin(\beta)}$$

(4) Note that $\beta = \pi - (\alpha + \gamma)$. Now,

$$\frac{\sin(\gamma) + \sin(\alpha)}{\sin(\beta)} = \frac{2\sin(\frac{a+\gamma}{2})\sin(\frac{a+\gamma}{2})}{2\sin(\beta/2)\cos(\beta/2)} = \frac{\cos(\beta/2)\cos(\frac{a+\gamma}{2})}{\sin(\beta/2)\cos(\beta/2)} \leq \frac{1}{\sin(\beta/2)}$$

(5) Now

$$\frac{AB + BC}{AC} \leq \frac{1}{\sin(\beta/2)} < \pi/\beta < \frac{2\pi - \beta}{\beta} = \frac{\pi}{\beta} + \frac{\pi - \beta}{\beta}$$

(6) From the previous part, $\beta(AB + BC) \leq AC(2\pi - \beta)$ so

$$2\pi(AB + BC) + \beta(AB + BC) \leq 2\pi(AB + BC + AC) - \beta AC$$

Thus,

$$\frac{AB+BC}{2\pi-\beta}\leq\frac{AB+BC+AC}{2\pi}$$

(7) Applying the previous part to triangle AOB,

$$\frac{AB+AO+OB}{2\pi}>\frac{AO+OB}{2\pi-\phi}\iff AB(2\pi-\phi)-\phi(AO+OB)>0$$

Thus with $\pi<2\pi-\phi$,

$$AB>\frac{\phi}{2\pi-\phi}(AO+OB)>\frac{\phi}{\pi}(AO+OB)$$

and $CD>\frac{\phi}{\pi}(CO+OD)$

(8) Add the previous inequalities so that $AB+CD>\frac{\phi}{\pi}(AO+OB+CO+OD)=\frac{\phi}{\pi}(AC+BD)$. Similarly $BC+AD>\frac{\phi}{\pi}(AC+BD)$.

(9) From here we have $\pi(AB+CD+AC+BD)<(AB+CD+BC+AD)(\phi+\pi)$. It remains to divide by the angular parts.

Problem 8.(15=2+2+2+5+4)

(1) $K(\Gamma)=3\pi$ and $L(\Gamma)=2(AY+YD)$

(2) $K(\gamma)=3\pi+4\theta$

(3) $L(\gamma)=(AY+YD)(1+\sec(\theta)+\tan(\theta))$

(4) For this part note that

$$\frac{3\pi}{2}<\frac{3\pi+4\theta}{2+\tan\theta}\implies\frac{3\pi}{2(AY+YD)}>\frac{3\pi+4\theta}{(AY+YD)(1+\sec(\theta)+\tan(\theta))}$$

Hence it suffices to demonstrate that the left is satisfied. But the left is equivalent to,

$$6\pi+8\theta<6\pi+3\pi\tan\theta\iff\frac{8}{3\pi}<\frac{\tan\theta}{\theta}$$

(5) With the unit circle draw the line $y=1$. Now consider a sector of angle 2θ centered on the line $x=0$ and extend the sector radii until they intersect the line $y=1$. Note that the sector has area 2θ and the triangle containing it has area $\tan(\theta)$ hence, $\tan(\theta)>\theta$ or $\frac{\tan(\theta)}{\theta}>1$ Now $3\pi>9$ so $\frac{8}{3\pi}<1$.

Problem 9.(5) The curvature at a vertex is 2π minus the sum of vertex angles. But the faces are equilateral triangles hence, $K=2\pi-(\pi/3)3=\pi$.

Problem 10.(10)
By Theorem 6.2, we know that an icosahedron has F=20, n=12, E=30 so the total curvature is $2\pi(12+20-30)=4\pi$. But icosahedrons are convex so each vertex contributes $4\pi/12=\pi/3$ curvature. Also notice that a convex polyhedron has curvature 4π, so it follows that the dodecahedron has curvature $4\pi/20=\pi/5$ at each vertex.

Problem 11.(25)
Proceed as in problem 7. At one face of B extend the face into a plane that intersects A, now make the modification that the section of A contained within the boundary of the intersection is moved down to the plane call this section of A $\beta(A)$. Clearly the absolute curvature is still 4π due to convexity. It remains to show that the area is less, then proceed in this fashion on all other sides to complete the proof, the final figure will be exactly the convex hull of B which is B.

To prove that the area decreased, notice that the projection of each edge of $\beta(A)$ onto the plane generates a natural partition of the plane into polygons. It follows that for a given n-gon in the plane (P), the corresponding region in $\beta(A)$ is an n-gon with area $\frac{area(P)}{\cos\theta}$ where θ is the angle between the normal to the n-gon in $\beta(A)$ and the normal to the plane. It follows that the area in the plane is always less than or equal to that in $\beta(A)$.

Problem 12.(**10**) In this case the DNA inequality is equivalent to Problem 11, because γ convex means that both γ and Γ are convex and have curvature 4π.

Problem 13.(**25=12+13**)
This problem is hard. Here one solution is provided, but there are potentially other possibilities.

a) Use the following type of construction, we specify polyhedra with the property that any line perpendicular to the axis of rotation intersects the surface only twice and such that the surface has rotational symmetry.

b)Claim: It follows now that the convex hull of this surface has the property that the surface area is not less than that of the surface. Proving this is sufficient because $K(\Gamma) = 4\pi$, $K(\gamma) \geq 4\pi$ so the DNA inequality will follow.

Notice that each segment in the cross section generates a conical frustum when rotates around the figure, but the convex hull generates a conical frustum that contains the inner one. Because the area is $\pi r h$ and h is the same for both frusta, the outer one must have greater area. It follows that the convex hull has greater area. QED