12^{th} Annual Johns Hopkins Math Tournament
Saturday, February 19, 2011
Explorations Unlimited Round-The DNA Inequality

1. Introduction

The DNA inequality is a geometric inequality that relates the average of curvature two curves in space, one that is contained inside the other. The DNA inequality has been proved for two dimensions but the general form of the DNA inequality in n-dimensional space remains unproven. In this round we will guide you through what the inequality means and some interesting corollaries and counterexamples.

The DNA inequality gets its name from the similarity of the concepts to the way we might draw DNA in a cell nucleus. It just comes from the pictures of the curves that we think about. In order to explain these pictures we really need to state the inequality in a formal way and then explain what we mean so here it is...

Theorem 1.1. *[The DNA Inequality] For any closed curve, γ, completely contained within a larger closed convex curve, Γ, the average absolute curvature of γ is always greater than or equal to the average absolute curvature of Γ.*

2. Closed Convex Curves

You may already have an idea of what Theorem 1.1 means but there some very important terms in this theorem that warrant discussion.

First of all γ and Γ are **closed** curves. This means that they don't have a beginning or end. If you trace one with a pencil you end up where you started without ever lifting it. They are essentially closed loops.

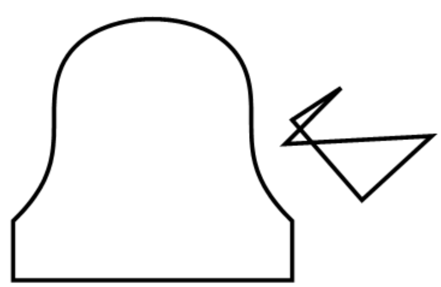


Figure 1. Closed Plane Curves

Definition 2.1. A closed curve is a curve with no endpoints and which completely encloses an area in the plane.

Next, Γ is not only closed but also **convex**. This is a bit trickier. Formally this means that if I draw a line between any two points on the curve it will be completely inside the curve. Intuitively this means that there are no indentations or crossings, the curve cannot be concave anywhere. Circle, Ellipses, and rectangles are all examples of convex curves (No one said it had to be smooth!) Rigorously we first define convex sets,

Definition 2.2. A set is convex if for every pair of points within the set, every point on the straight line segment that joins them is also within the set.

Definition 2.3. A convex curve is the boundary of a convex set.

Before we go on, let's spend a bit more time on the notions of convexity in the following problems...

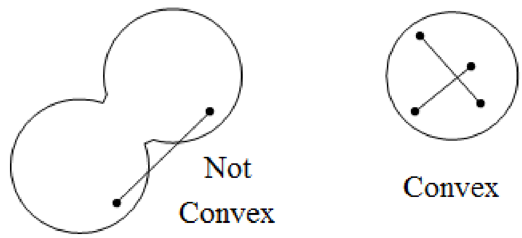


Figure 2. Convexity in the Plane

Problem 1 (The Happy Ending Problem). (**20=5+15**) Erdos once proposed the following problem: Given a set of points in the plane such that no three are collinear, how many are required to ensure that for every configuration there are n points that form a convex n-gon? We call this number $g(n)$. Clearly for $n = 3$ $g(n) = 3$, that is, any three non-collinear points form a triangle which is the convex 3-gon.

(1) What is $g(4)$? There are 3 cases for the orientations of the points.
(2) It has been shown that $g(n)$ increases exponentially making computations difficult. Prove that $2^{n-2} \leq g(n)$.

Problem 2 (Helly's Theorem). (**12=4+4+4**) Suppose you have a collection of 6 non-empty convex sets in $\mathbb{R}$. These sets are intervals in the number line. Suppose the sets have the property that any selection of n of them has a non-empty intersection, where n is some non-negative integer.

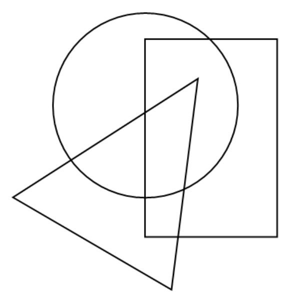


Figure 3. Helly's Theorem

(1) What is the minimum n such that the intersection of all the sets is non-empty?
(2) Suppose I now consider another collection of 8 non-empty convex sets in $\mathbb{R}^2$ with the same property. What is the minimum value of n?
(3) Consider any finite collection of non-empty sets in $\mathbb{R}^3$. What is the minimum value of n?

3. Polygonal Curves

Approaching a statement about smooth curves such as Theorem 1.1 can be a daunting task. We consider instead a formulation in terms of polygonal curves. Let us begin with a definition,

Definition 3.1. A polygonal curve is a curve specified by a sequence of points called its vertices so that the curve consists of the line segments connecting the consecutive vertices.

If we only consider polygonal curves the geometric notion of curvature becomes much more easily expressed. You can think of curvature as the amount the curve has to turn, we measure it in radians. This means how much is it NOT like a straight line, which has 0 curvature. The **curvature** of the curve is completely contained in each vertex of the curve and is formally defined to be the exterior angle θ_i between two segments of the curve joined to that vertex. The **total curvature** for the entire curve is simply a sum over the vertices

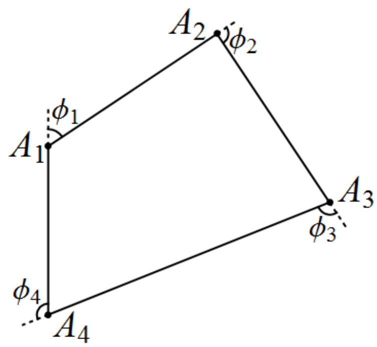


FIGURE 4. Plane Ploygonal Curve

of all the exterior angles, $\sum \theta_i$. Likewise, summing the absolute value of each angle we obtain the **absolute curvature** of the curve, $\sum |\theta_i|$. Finally the **average absolute curvature** is defined to be the absolute curvature of a curve divided by its length.

Let us compute the total curvature of a square. We can immediately see that the curvature at each vertex is 1/4 of a circle or $\pi/2$ radians. Hence the absolute curvature is equal to the total curvature and is 2π. For our square example, supposing the side length is 2 then the absolute average curvature is $\frac{2\pi}{2 \cdot 4}$.

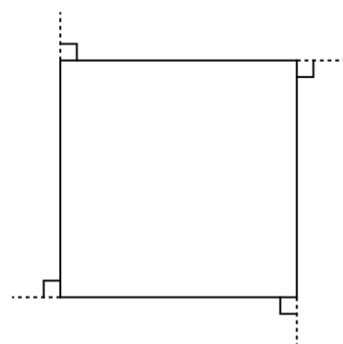


FIGURE 5. Square Curve

There is one additional subtle point that is not illustrated above, curvature is **signed**. That is, if the curve turns right by $\pi/8$ and then left $\pi/8$ the total curvature is 0! We end up in the same direction as when we started. This means we need to assign a different sign to curvature leftwards and rightwards, and we must consistently "travel" in only one direction when doing the computation. It happens, though not by accident, that the total curvature of a circle is 2π. To see this suppose you move a tangent segment around a circle, in the figure. The tangent line turns through every angle between 0 and 2π only once, in making a complete rotation.

Problem 3. (**10**)Prove that the total curvature of a closed polygonal curve is always a multiple of 2π.

Now that you have some idea about curvature we can define average absolute curvature. This just means we take the absolute value of the curvature at every point and after we are done adding it up for the entire curve we divide by the length of the curve to get an average.

Problem 4. (**12=6+6**) Compute the average absolute curvature of the following curves assuming they both have length 1.

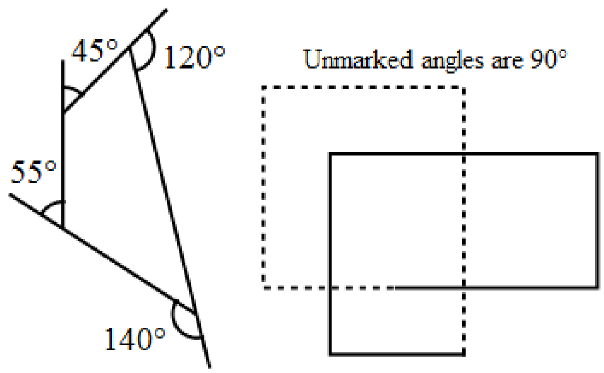
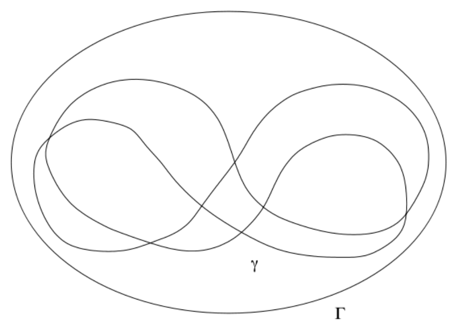


FIGURE 6. Problem 4

So the resulting image for cases of Theorem 1.1 looks like a convex shape enclosing a jumbled curve. This is very much like DNA in a cell nucleus, hence the name.



3.1. **Polygonal Approximations.**

Our definition of curvature in terms of polygonal curves was no mistake. Although the problem of the DNA inequality is fundamentally one of all curves the case for polygonal curves provides not only a more elementary proof but it is good enough for the smooth case as well! This is because of the following fact:

Theorem 3.2. *Given any smooth curve there is a polygonal line that approximates it to within any fixed positive error.*

That is, we can get *as close as we want* to a smooth curve with a polygonal one![1]This may seem trivial but it is a very important fact. Not only is it the basis for the proof of Theorem 1.1 but also in other applications in approximation of smooth functions.

We are now ready for an **equivalent** approachable restatement of the theorem:

Theorem 3.3. *For any closed polygonal curve, γ, completely contained within a larger closed convex polygonal curve, Γ, the average absolute curvature of γ is always greater than or equal to the average absolute curvature of Γ.*

[1]Here we don't explain what "error" means precisely, nor do we explain "as close as we want." More precisely, for all $\varepsilon > 0$ the maximum of the set of maximum distances between a segment and all points on the curve between the endpoints of the segment is defined as the error and is less than ε.

4. The Two Dimensional Case

4.1. **The Circle.**

The first easy case of the theorem is when we let Γ be the unit circle. As we travel all the way around a circle we end up in the same direction that we started (only at the end) but we only turned one direction hence the curvature of a circle is 2π as mentioned previously. You will now prove the DNA inequality for polygonal curves contained in this circle:

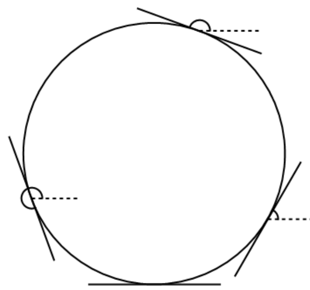


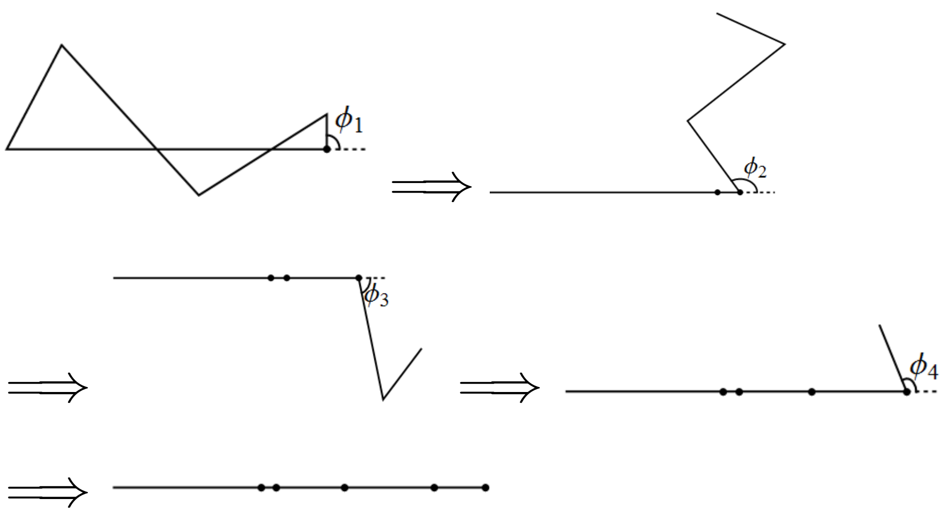
Figure 7. Curvature Around a Circle

Theorem 4.1. *For any closed polygonal curve, γ, completely contained within the unit circle, the average absolute curvature of γ is always greater than or equal to that of the unit circle.*

Problem 5 (Proof of Theorem 3.1)**.** (**18=2+2+4+4+3+3**)

(1) Let the absolute curvature of γ be $K(\gamma)$ and let its length be $L(\gamma)$. What is the average absolute curvature of the unit circle? Of γ?
(2) Show that the theorem is equivalent to saying that the curvature of γ is not less than its length.
(3) Suppose we were to "unfold" γ into a line by rolling it over the vertices. What we mean by this is, instead of traveling around the curve as we did in the previous section we will straighten it out into a line and keep track of how it moves.

    Start with one side on the x-axis, now one of the adjacent segments makes an angle with the x-axis. Rotate the curve and the entire plane along this angle until this segment is also aligned with the x-axis. As you rotate keep the straight part fixed. Let us call the end point of the last segment A.

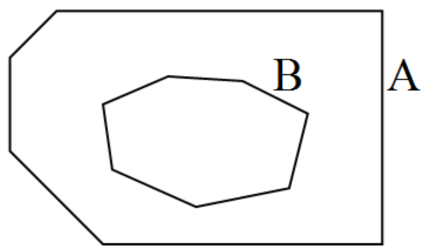


What is the distance between the initial and final positions of A after the unfolding?

(4) Now consider the trajectory of the circle's center, O. As we unfold the polygon, O travels in arcs based on the exterior angles of the polygon. What is the largest radius of one of these arcs and what is its length if it corresponds to the exterior angle φ_i?
(5) Find the total length of the trajectory of O?
(6) Complete the proof of the theorem.

4.2. **Lagarias and Richardson.**
Lagarias and Richardson were the first to propose a proof for the two dimensional case of Theorem 3.3. They analyzed polygonal lines and considered the cases of non-convexities in their boundaries.

Problem 6. (**20**) Let A and B be closed convex polygonal curves in the plane such that A completely contains B. Prove that the average absolute curvature of B is greater than or equal to that of A (using the triangle inequality and induction on the number of sides of B.)



In the problem above you just proved that a convex curve contained inside another has the higher average absolute curvature, but in the case of Theorem 3.3, Γ has less average absolute curvature than γ. Hence, if it is true for the smallest convex curve containing γ, it must also be true for all those containing it. See the figure.

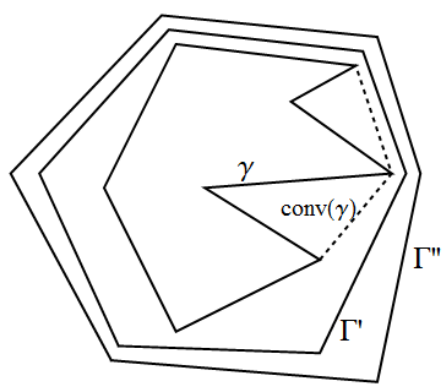


FIGURE 8. Convex Hull

The "smallest" convex curve containing a non-convex closed curve, γ, is called the convex hull of γ or $conv(\gamma)$. You can think of this as putting a rubber band around the outside of γ. It will shrink as much as possible but will never curve in on itself. Formally this is the minimal convex set containing the curve. For notation let $conv(\gamma)$ be the convex hull of the curve γ

Based on the problem above, if we prove Theorem 3.3 for the case that $\Gamma = conv(\gamma)$ this will be sufficient to prove all of the Theorem!

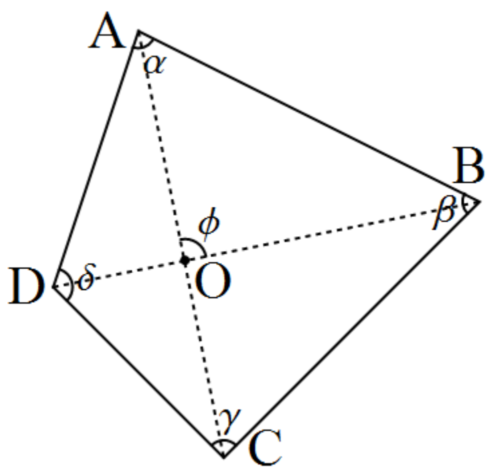
Lagarias and Richardson complete this proof by transforming the non-convex curve into its convex hull and demonstrating that the inequality holds after each transformation. Suppose that the average absolute curvature of a curve γ is $\mathcal{K}(\gamma)$. After a transformation of the inner curve γ call the new curve γ_1. They show that if the average absolute curvature only decreases with the transformations, then the theorem is satisfied. Explicitly,

$$\mathcal{K}(\gamma) \geq \mathcal{K}(\gamma_1) \geq \mathcal{K}(\gamma_2) \geq \ldots \geq \mathcal{K}(\Gamma)$$

Then clearly the average absolute curvature of the inner curve γ is greater than that of the outer curve Γ.

Let's begin with a result for quadrilaterals...

Problem 7. (**18=2×9**) For any convex quadrilateral $ABCD$ there are 2 possible non-convex ones, $ABDC$ and $ACBD$. Consider $ABDC$ in the diagram to the right. We want to show that if we "untangle" the diagonals of $ABDC$ into $ABCD$(its convex hull) the DNA inequality holds.



(1) First write down the average absolute curvature of $ABCD$.
(2) Write down the average absolute curvature of $ABDC$, and the DNA inequality.
(3) Prove that:

$$\frac{AB + BC}{AC} = \frac{\sin\alpha + \sin\gamma}{\sin\beta}$$

(4) Prove that:

$$\frac{\sin\alpha + \sin\gamma}{\sin\beta} = \frac{\cos\left(\frac{\alpha+\gamma}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \leq \frac{1}{\sin\left(\frac{\beta}{2}\right)}$$

(5) We know from the concavity of the sine function that $\sin x/2 < \frac{x}{\pi}$ on $[0, \pi/2]$. Prove that

$$\frac{AB + BC}{AC} < \frac{2\pi - \beta}{\beta}$$

(6) Prove that Problem 7 Part (3) is equivalent to:

$$\frac{AB + BC}{2\pi - \beta} < \frac{AB + BC + AC}{2\pi}$$

(7) Let ϕ be the angle $\angle AOB$. Show that:

$$AB > \frac{\phi}{\pi}(AO + OB)$$

$$CD > \frac{\phi}{\pi}(CO + OD)$$

(8) Use Problem 7 Part (5) to show that $AB + CD > \frac{\phi}{\pi}(CD + BD)$ Similarly conclude that $BC + AD > \frac{\phi}{\pi}(AC + BD)$.

(9) It now follows that:

$$\frac{\phi}{\pi}\frac{AC+CD}{AC+BD}+(1+\frac{\phi}{\pi})\frac{BC+AD}{AC+BD}>\frac{\phi}{\pi}\frac{\phi}{\pi}+(1+\frac{\phi}{\pi})(1-\frac{\phi}{\pi})=1$$

Hence,

$$\frac{AC+CD}{AC+BD}+1<(1+\frac{\phi}{\pi})(\frac{AC+CD}{AC+BD}+\frac{BC+AD}{AC+BD})$$

Complete the proof of the DNA inequality for the quadrilateral ABCD.

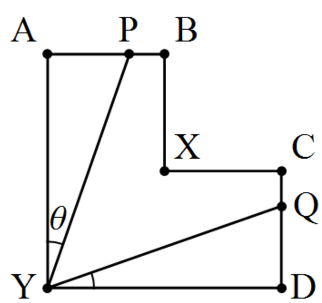
The result in the problem above is actually quite significant, and can be expanded as follows. If the inner curve were more complicated than the quadrilateral case but one of the differences between it and the convex hull was a "twist" as we saw in the diagonals of the quadrilateral, then it can be "untwisted." That is, I can make the improvement to untwist the diagonals and the DNA inequality holds!

The remainder of the proof deals with cases like that above. They document every sort of tangle or intersection the inner curve may exhibit. They demonstrate that in each case the curve can be improved to its convex hull via transformations that preserve the inequality. This method is known as "brute force" and unfortunately remains the only proof of the theorem.

5. What About Non-convex Curves?

In Lagarias and Richardson's paper it was suggested that the DNA inequality might also be true for some non-convex outer domains, in particular L-shapes. An L-shape is the type of shape that results when you take a square and remove a smaller square from the corner. Eric Larson proved this conjecture to be false. Here we guide you through the proof...

Problem 8. **(15=2+2+2+5+4)** We proceed by constructing a counterexample. Consider the curve $\gamma = APYQDYA$ inside the L-Shape $ABXCDY(=\Gamma)$. Choose θ sufficiently small and construct P and Q according to θ.



Now we want to prove that the DNA inequality is false so we need to show that the reverse of the inequality is true for this curve. That is,

$$\frac{K(\Gamma)}{L(\Gamma)}>\frac{K(\gamma)}{L(\gamma)}$$

(1) Compute $K(\Gamma)$ and $L(\Gamma)$.
(2) Compute $K(\gamma)$.
(3) Compute $L(\gamma)$.
(4) Prove that the inequality **above** reduces to

$$\frac{8}{3\pi}<\frac{\tan\theta}{\theta}$$

(5) Now we must prove that the result from part 4 is indeed correct. Prove that

$$\frac{8}{3\pi} < 1 < \frac{\tan\theta}{\theta}$$

(A geometric proof of this fact will receive more points than one from calculus.) This shows that the RHS is always greater than the LHS. Hence the opposite of the DNA inequality is true, making this a valid counterexample. This completes the proof.

6. Generalizations

6.1. Curvature in Three Dimensions.

One might imagine that in 3 dimensional space the curves Γ and γ of the DNA inequality could take the form of surfaces and 3D lines. It is by this method that we generalize Theorem 1.1. There are two different cases: γ can either be a curve or a surface. For the purposes of this exam we take γ to be a surface. We might generalize Theorem 1.1 as follows:

For any surface, γ, completely contained within a larger closed convex surface, Γ, the average absolute curvature of γ is always greater than or equal to the average absolute curvature of Γ.

But we would quickly discover that the term "curvature" is not well-defined in more than two dimensions. We would also be at a loss when attempting to compute the *average* part of our curvatures. Again, for a more approachable problem we consider the polygonal analogue of surfaces. These are polyhedra.

In the case of polyhedra, like polygonal curves, the curvature is contained in the vertices.

Definition 6.1. The curvature at the vertex of a closed polyhedron is 2π minus the sum of the vertex angles.

From this definition, it follows that the total curvature and absolute curvature are computed by summing the curvatures over all the vertices of the polyhedron. The average absolute curvature is then defined by dividing by the total surface area of the polyhedron.

In the spirit of our two dimensional square example, let us proceed with a cube. The cube has 8 identical vertices each adjoining 3 faces. The angles of the corners of each square face are $\pi/2$ at the vertex. Hence the curvature at one vertex is $2\pi - 3(\pi/2) = \pi/2$. This implies that the total curvature of the cube is $8(\pi/2) = 4\pi$.

Problem 9. (**5**) Consider the tetrahedron. What is the curvature at each vertex?

As it turns out, similar to the 2 dimensional case, the total curvature of **any convex surface** is 4π. From this it follows immediately that the absolute curvature of any surface is not less than 4π.

On the subject of polyhedral curvature, there is a miraculous theorem due to Gauss that states:[2]

Theorem 6.2. *Let a closed polyhedron with vertices $\{a_1, a_2 \ldots a_n\}$ have curvature at each vertex given by $\mathcal{K}(a_i)$. Let the number of faces be F and the number of edges be E, then*

$$\sum_{1}^{n} \mathcal{K}(a_i) = 2\pi(n - E + F)$$

Thankfully, the notion of convexity naturally extends to 3 dimensions. In order for a polyhedron to be convex, line segments with ends on the convex polyhedron must be completely contained within the polyhedron or one of its faces. The analogy of the rubber-band extends to a rubber balloon stretched over a non-convex

[2]Precisely, we need the polyhedron to be homeomorphic to a sphere.

polyhedron.[3]

Problem 10. (**10=5+5**) What is the curvature at each vertex of a regular icosahedron? a regular dodecahedron?

6.2. **Cylindrical Symmetry.**
Unfortunately there are very few results for the three dimensional case of the theorem. An analogous version of the theorem for surfaces inside spheres (much like our proof for curves in circles) is known, but it is not readily expanded into a full proof. Here we will guide you through a few simple results that are known.

First of all, let us provide a re-statement of Theorem 1.1 in 3 dimensions..

Definition 6.3. The absolute curvature of a polyhedral surface is the sum over all vertices of the difference: $|2\pi - \sum \alpha_i|$, where α_i are the vertex angles.

Definition 6.4. The average absolute curvature of a polyhedral surface is the absolute curvature of the surface divided by its surface area.

Problem 11. (**25**) Let A and B be closed convex polyhedral surfaces in the $\mathbb{R}^3$ such that A completely contains B. Prove that the average absolute curvature of B is greater than or equal to that of A.

Theorem 6.5 (DNA-Inequality for Surfaces)**.** *For any polyhedral surface, γ, completely contained within a larger closed convex polyhedral surface, Γ, the average absolute curvature of γ is always greater than or equal to the average absolute curvature of Γ.*

Problem 12. (**10**) Show that when γ is convex the DNA inequality in 3 dimensions holds.

We end with a problem that reaches beyond the polyhedral scope of this section...

Problem 13. (**25=12+13**) A rotationally symmetric surface is one such that rotations of a fixed amount $< 2\pi$ about an axis leave the surface unchanged. For example a cube has rotation symmetry about a central axis of $\pi/2$ radians and a cylinder has rotation symmetry of any finite angle about its central axis. The DNA inequality holds for rotationally symmetric surfaces that satisfy certain restrictions.

(1) Find a set of surfaces such that the DNA inequality holds. (Hint: consider "semi-smooth" surfaces generated by rotating about an axis)
(2) Prove the DNA inequality in 3 dimensions for this class of surfaces.

[3]This is not strictly true. For certain polyhedra the convex hull is not congruent to the minimal energy surface containing the polyhedron.