12^{th} Annual Johns Hopkins Math Tournament
Saturday, February 19, 2011

Combinatorics/Probability Subject Test

1. [**1025**] Natasha flips 10 fair coins and counts the number of heads. What is the probability that Natasha flipped 5 heads and 5 tails?
Answer: $\boxed{\frac{63}{256}}$ The probability of any coin landing heads or tails is $\frac{1}{2}$. If we label our coins distinctly, then the probability of getting any particular combination is $\frac{1}{2^{10}}$. Since we do not care about the ordering of the coins, there are 10! ways to rearrange them divided by 5! ways to rearrange the heads and 5! ways to rearrange the tails. Hence, the probability is $\frac{1}{2^{10}}\frac{10!}{5!5!} = \frac{63}{256}$.

2. [**1026**] Find the number of pairs (a, b) with a, b positive integers such that $\frac{a}{b}$ is in lowest terms and $a + b \leq 10$.
Answer: $\boxed{32}$ If $\frac{a}{b}$ is in lowest terms, then it means that a and b are relatively prime; i.e. their greatest common divisor is 1. Now, a and b are relatively prime if and only if a and $a + b$ are relatively prime (indeed, if a number divides a and $a + b$ then it must also divide b). So now we just need to count the number of integers relatively prime to each of $1, 2, \ldots, 10$, which is given by

$$\sum_{i=1}^{10} \varphi(i) = 1 + 1 + 2 + 2 + 4 + 2 + 6 + 4 + 6 + 4 = 32,$$

where $\varphi(n)$ is the Euler-phi function.

3. [**1028**] An *icosahedron* is a regular polyhedron with 12 vertices, 20 faces, and 30 edges. How many rigid rotations G are there for an icosahedron in $\mathbb{R}^3$?
Answer: $\boxed{60}$ There are 12 vertices, each with 5 neighbors. Any vertex and any of its neighbors can be rotated to any other vertex-neighbor pair in exactly one way. There are $5 \cdot 12 = 60$ vertex-neighbor pairs.

4. [**1032**] You are standing at the base of a staircase with 11 steps. At any point, you are allowed to move either 1 step up or 2 steps up. How many ways are there for you to reach the top step?
Answer: $\boxed{144}$ Consider standing at the nth step. Let F_n denote the number of ways that you could have reached the nth step. There are two ways you could have reached this step: by taking one step up from the $n-1$st step, or by taking two steps up from the $n-2$nd step. Therefore, F_n is completely determined by $F_{n-1}, F_{n-2}, \ldots, F_1$. In particular, note that F_n is simply the number of ways to reach the $n-1$st step plus the number of ways to reach the $n-2$nd step: $F_n = F_{n-1} + F_{n-2}$. This is the recursive definition of the Fibonacci sequence. Here, the base cases are $F_1 = 1$, $F_2 = 2$ so we conclude that the number of ways to reach the 11th step is simply $F_{11} = 144$ (this can be quickly calculated by hand).

5. [**1040**] Mordecai is standing in front of a 100-story building with two identical glass orbs. He wishes to know the highest floor from which he can drop an orb without it breaking. What is the minimum number of drops Mordecai can make such that he knows for certain which floor is the highest possible?
Answer: $\boxed{14}$ Consider dropping the orb from the nth floor. If the orb breaks, then we should go down to the lowest floor from which we know it will not break. In this case, that would be ground level so go to the first floor and drop the second orb. If it breaks, we are done. Otherwise, we go up to the second floor and continue. In this case, it will take no more than n drops to find the desired floor. Now, suppose that the orb did not break when dropped from the nth floor. Go up to floor $n + k$. If the orb breaks, go to floor $n+1$ (because we know it won't break on floor n). If it breaks, we're done; otherwise, proceed to $n + 2$ and proceed as before. In this case, it will take at most $2 + k - 1 = k + 1$ drops. But this should not require any more drops than the first time, so we have $n = k + 1$, or $k = n - 1$. Now, if the orb did not drop on the $n + k$th floor, proceed up to the $n + k + \ell = 2n + \ell - 1$th floor. Repeat the process. We can conclude that $3 + \ell - 1 = n$, or $\ell = n - 2$. Continuing inductively, we will ultimately end up on floor $n + (n-1) + (n-2) + \cdots + 1 = \frac{n(n+1)}{2}$ (assuming the orb never broke). The desired n is the smallest one such that $\frac{n(n+1)}{2} > 100$, because there are 100 floors. This is easily computed to be $n = 14$.

6. [**1056**] Two ants, Yuri and Jiawang, begin on opposite corners of a cube. On each move, they can travel along an edge to an adjacent vertex. Find the probability they both return to their starting position after 4 moves.

Answer: $\boxed{\dfrac{49}{729}}$ Let the cube be oriented so that one ant starts at the origin and the other at $(1,1,1)$. Let x, y, z be moves away from the origin and x', y', z' be moves toward the origin in each the respective directions. Any move away from the origin has to at some point be followed by a move back to the origin, and if the ant moves in all three directions, then it can't get back to its original corner in 4 moves. The number of ways to choose 2 directions is $\binom{3}{2} = 3$ and for each pair of directions there are $\frac{4!}{2!2!} = 6$ ways to arrange four moves a, a', b, b' such that a precedes a' and b precedes b'. Hence there are $3 \cdot 6 = 18$ ways to move in two directions. The ant can also move in a, a', a, a' (in other words, make a move, return, repeat the move, return again) in three directions so this gives $18 + 3 = 21$ moves. There are $3^4 = 81$ possible moves, 21 of which return the ant for a probability of $\frac{21}{81} = \frac{7}{27}$. Since this must happen simultaneously to both ants, the probability is $\frac{7}{27} \cdot \frac{7}{27} = \frac{49}{729}$.

7. [**1088**] If a rectangle is drawn in a 2011×2011 square grid (degenerate rectangles do not count), what is the expected value of the area of the rectangle?

Answer: $\boxed{671^2}$ Suppose the grid is contained within the square $(0,0)$, $(2011,0)$, $(0,2011)$, $(2011,2011)$. The expected value of the area of the rectangle X is

$$\mathbb{E}([X]) = \frac{\sum_{i=1}^{n}[X_i]}{n}$$

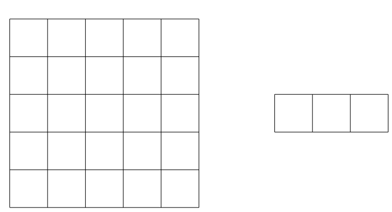
where $[X_i]$ denotes the area of the ith rectangle. We first determine the number of possible rectangles. A rectangle is completely determined by the two vertical lines and two horizontal lines; there are $\binom{2012}{2}$ choices for the two vertical lines and similarly for the horizontal, for $\binom{2012}{2}^2$ total rectangles. Next we need the sum of all of these areas. In choosing a horizontal length ℓ, we can use any of the intervals $[0,\ell]$, $[1,\ell+1], \ldots, [2011-\ell, 2011]$, for a total of $2012-\ell$ choices. Similarly, for a height of h there are $2012-h$ possible choices. We know the area of such a rectangle is ℓh. Therefore, the sum of all the areas is

$$\begin{aligned}
\sum_{\ell=1}^{2011}\sum_{h=1}^{2011}(2012-\ell)(2012-h)\ell h &= \left(\sum_{\ell=1}^{2011}(2012-\ell)\ell\right)\left(\sum_{h=1}^{2011}(2012-h)h\right) \\
&= \left(\sum_{\ell=1}^{2011} 2012\ell - \sum_{\ell=1}^{2011}\ell^2\right)^2 \\
&= \left(1006 \cdot 2011 \cdot 2012 - \frac{2011 \cdot 2012 \cdot 4023}{6}\right)^2 \\
&= (1006 \cdot 2011 \cdot 2012 - 1006 \cdot 1341 \cdot 2011)^2 \\
&= (671 \cdot 1006 \cdot 2011)^2
\end{aligned}$$

Now to compute the expected value, we just divide this by the number of possible rectangles which we found above, since each rectangle is equally likely to be chosen. Hence the expected value is

$$\mathbb{E}([X]) = \frac{(671 \cdot 1006 \cdot 2011)^2}{\binom{2012}{2}^2} = \frac{(671 \cdot 1006 \cdot 2011)^2}{(1006 \cdot 2011)^2} = 671^2.$$

8. [**1152**] Consider the following 5-by-5 square and 3-by-1 rectangle:



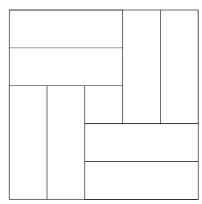
Define a *tiling* of the square by the rectangle to be a configuration in which eight nonoverlapping 3-by-1 rectangles are placed inside the 5-by-5 square, possibly rotated by 90 degrees but with grid lines matching up, with only one subsquare of the 5-by-5 square remaining uncovered. Find the number of such tilings, counting rotations and reflections as distinct.

Answer: $\boxed{2}$ First we will show that every possible tiling must leave the center subsquare uncovered. We number the subsquares of the 5-by-5 square in two different ways:

1	2	3	1	2
2	3	1	2	3
3	1	2	3	1
1	2	3	1	2
2	3	1	2	3

2	1	3	2	1
3	2	1	3	2
1	3	2	1	3
2	1	3	2	1
3	2	1	3	2

In each diagram, there are eight 1's, nine 2's, and eight 3's. Since every 3-by-1 rectangle within either square consists of subsquares labeled 1, 2, and 3, eight nonoverlapping 3-by-1 rectangles must cover eight of each, and therefore must leave a 2 uncovered. The only square labeled 2 in both diagrams is the center subsquare, so this subsquare must be uncovered in any tiling of the square. Now every other subsquare must be covered by one of the rectangles, so we can begin placing rectangles. Without loss of generality, we can assume that the subsquare in the top-left corner belongs to a horizontally oriented rectangle. This leaves two subsquares in the top row, which must both belong to vertically oriented rectangles. Rotating the figure and continuing, the following tiling is forced:



So this configuration and its mirror image are the only two possible tilings.

9. [**1280**] Determine the maximum number of ways that 10 circles and 10 lines can divide the plane into disjoint regions.

   Answer: $\boxed{346}$ Any arrangement of 10 lines and 10 circles can be constructed in any order. Ten lines such that no two are parallel and no three have a common intersection divide the plane into $1 + (1 + 2 + \cdots + 10) = 56$ regions. Each new circle creates additional regions equal in number to the number of new points of intersection between itself and the other lines and circles. Then we know that the answer is $56 + I$ where I is the number of intersections between two shapes such that at least one is a circle. Since a circle can intersect a line in up to two places, there are at most $2 \cdot 10 \cdot 10 = 200$ circle-line intersections. Two circles intersect in at most 2 points so there are at most $2 \cdot \binom{10}{2} = 90$ circle-circle points of intersection. Hence the maximal possible number is $56 + 200 + 90 = 346$.

10. [**1536**] How many functions f that take $\{1, 2, 3, 4, 5\}$ to itself, not necessarily injective or surjective, satisfy $f(f(f(x))) = f(f(x))$ for all x in $\{1, 2, 3, 4, 5\}$?

    Answer: $\boxed{756}$ For any such function f, let $A = \{n \mid f(n) = n\}$ be the set of elements fixed by f and let $B = \{n \mid f(n) \in A \text{ and } n \notin A\}$ be the set of elements that are sent to an element in A, but are not themselves in A. Finally, let $C = \{1, 2, 3, 4, 5\} \setminus (A \cup B)$ be everything else. Note that any possible value of $f(f(x))$ is in A so A is not empty. We will now proceed by considering all possible sizes of A.

    (a) A has one element: Without loss of generality, let $f(1) = 1$, so we will mutliply our result by 5 at the end to account for the other possible values. Suppose that B has n elements so C has the remaining $4 - n$ elements. Since $f(f(x)) = 1$ for each x so any element c in C must satisfy $f(c) = b$ for some b in B, because $f(c) \neq 1$ and the only other numbers for which $f(x) = 1$ are the elements of B. This also implies that B is not empty. Conversely, any function satisfying $f(c) = b$ works, so the total number of functions in this case is $5 \sum_{n=1}^{4} \binom{4}{n} n^{4-n}$ because there are $\binom{4}{n}$ ways to choose the elements in B, and each of the $4 - n$ elements in C can be sent to any element of B

(there are n of them). This sum is equal to $5(4+6\cdot 4+4\cdot 3+1)=205$, so there are 205 functions in this case that A has one element.

(b) A has two elements: This is similar to the first case, except that each element in B can now correspond to one of two possible elements in A, so this adds a factor of 2^n. The sum now becomes $\binom{5}{2}\sum_{n=1}^{3}\binom{3}{n}2^n n^{3-n} = 10(3\cdot 2+3\cdot 4\cdot 2+8)=380$, so there are 380 functions in this case.

(c) A has three elements: This is again similar to the prior cases, except there are 3 possible targes in A, adding a factor of 3^n. Then the sum is $\binom{5}{3}\sum_{n=1}^{2}\binom{2}{n}3^n n^{2-n} = 10(2\cdot 3+9)=150$, so there are 150 functions in this case.

(d) A has four elements: The logic is the same as the prior cases and there are $5(4)=20$ functions in this case.

(e) A has five elements: The identity function is the only possible function in this case.

Adding together the five cases, we see that there are $205+380+150+20+1=756$ such functions.