12^{th} Annual Johns Hopkins Math Tournament
Saturday, February 19, 2011

Calculus Subject Test

1. [**1025**] If $f(x) = (x-1)^4(x-2)^3(x-3)^2$, find $f'''(1) + f''(2) + f'(3)$.
Answer: $\boxed{0}$ A polynomial $p(x)$ has a multiple root at $x = a$ if and only if $x - a$ divides both p and p'. Continuing inductively, the nth derivative $p^{(n)}$ has a multiple root b if and only if $x - b$ divides $p^{(n)}$ and $p^{(n+1)}$. Since $f(x)$ has 1 as a root with multiplicity 4, $x - 1$ must divide each of f, f', f'', f'''. Hence $f'''(1) = 0$. Similarly, $x - 2$ divides f, f', f'' so $f''(2) = 0$ and $x - 3$ divides f, f', meaning $f'(3) = 0$. Hence the desired sum is 0.

2. [**1026**] Evaluate the integral $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1 + (\tan x)^{\pi e}}$.
Answer: $\boxed{\dfrac{\pi}{4}}$ We make the substitution, $x = \frac{\pi}{2} - y$ (note that the actual variable of integration is irrelevant so we leave it as x). Then we have:
$$\int_0^{\frac{\pi}{2}} \frac{dx}{1 + (\tan x)^{\pi e}} = \int_{\frac{\pi}{2}}^0 \frac{-dx}{1 + \tan\left(\frac{\pi}{2} - x\right)^{\pi e}} = \int_0^{\frac{\pi}{2}} \frac{dx}{1 + (\cot x)^{\pi e}} = \int_0^{\frac{\pi}{2}} \frac{(\tan x)^{\pi e}\, dx}{(\tan x)^{\pi e} + 1}.$$
Then we add the original integral to both sides:
$$2\int_0^{\frac{\pi}{2}} \frac{dx}{1 + (\tan x)^{\pi e}} = \int_0^{\frac{\pi}{2}} \frac{1}{1 + (\tan x)^{\pi e}} + \frac{(\tan x)^{\pi e}}{1 + (\tan x)^{\pi e}} dx = \int_0^{\frac{\pi}{2}} \frac{1 + (\tan x)^{\pi e}}{1 + (\tan x)^{\pi e}} dx = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}.$$
So the integral we want is $\frac{\pi}{4}$.
Remark: The fact that the exponent is something absurd like πe suggests that it should be irrelevant to computing the integral. Therefore, the smart thing to do is to replace πe by something more managable, such as 0 or 2.

3. [**1028**] What is the minimal distance between the curves $y = e^x$ and $y = \ln x$?
Answer: $\boxed{\sqrt{2}}$ Observe that since the two curves are inverses of each other, they are symmetric about the line $y = x$. Therefore, it suffices to determine the minimum distance between $y = x$ and one of the curves, say $y = \ln x$. Fix a point (a, a) on the line $y = x$. The shortest distance between this point and the curve $y = e^x$ is given by the normal line at this point intersecting $y = e^x$ (indeed, the shortest distance between two points is a straight line). This line has equation $y = -x + 2a$ and intersects the curve when $e^x = -x + 2a$. Thus we seek to minimize the quantity
$$d^2 = (x-a)^2 + (e^x - a)^2 = (a - e^x)^2 + (e^x - a)^2 = 2(e^x - a)^2 = 2\left(\frac{e^x + x}{2}\right)^2.$$
This is minimized when
$$(e^x - x)(e^x - 1) = 0$$
As $x < e^x$ for all x for this to be zero we need $e^x - 1 = 0 \Rightarrow x = 0$. Thus the minimizing point on the curve is $(0, 1)$ and so the corresponding point on $y = x$ is $(a, a) = (\frac{1}{2}, \frac{1}{2})$. The distance between these points is $\frac{\sqrt{2}}{2}$ and so by considering symmetry, the distance between $y = e^x$ and $y = \ln x$ is $\sqrt{2}$.

4. [**1032**] Let f be one of the solutions to the differential equation
$$f''(x) - 2xf'(x) - 2f(x) = 0.$$
Supposing that f has Taylor expansion
$$f(x) = 1 + x + ax^2 + bx^3 + cx^4 + dx^5 + \cdots$$

near the origin, find (a, b, c, d).

Answer: $\boxed{\left(1, \frac{2}{3}, \frac{1}{2}, \frac{4}{15}\right)}$ We simply need to compute the Taylor expansion of $f''(x) - 2xf'(x) - 2f(x)$ to the third term, which is

$$(2a + 6bx + 12cx^2 + 20dx^3 + \cdots) - (2x + 4ax^2 + 6bx^3 + \cdots) - (2 + 2x + 2ax^2 + 2bx^3 + \cdots).$$

All coefficients should be zero, so $2a - 2 = 0$, $6b - 4 = 0$, $12c - 6a = 0$ and $20d - 8b = 0$. Solving these equations gives the desired coefficients.

5. [**1040**] How many real zeroes does the function $f(x) = \frac{x^{2011}}{2011} + \frac{x^{2010}}{2010} + \cdots + x + 1$ have?

Answer: $\boxed{1}$ Since $f(x)$ is of odd degree, by the Intermediate Value Theorem it has a real root a. We claim that this is the only root. Suppose that $b \neq a$ is also a real root. Then $f(a) = f(b) = 0$. Since $f(x)$ is a polynomial it is differentiable and so by Rolle's Theorem, there exists a point c in (a, b) such that $f'(c) = 0$. Now $f'(x) = x^{2010} + x^{2009} + \cdots + x + 1$. Then

$$(x-1)f'(x) = (x-1)\left(x^{2010} + x^{2009} + \cdots + x + 1\right) = x^{2011} - 1.$$

But 1 is the only real root of this function (the other roots are the other 2011th roots of unity) and that was contributed when we multiplied by $x - 1$. Hence $f'(x) \neq 0$ for any real x. But this contradicts Rolle's Theorem and so there must be only one real root.

6. [**1056**] Find the maximum value of a and minimum value of b such that $a \leq \frac{\arctan x}{x} \leq b$ for $0 \leq x \leq 1$. Express your answer as an ordered pair (a, b).

Answer: $\boxed{\left(\frac{\pi}{4}, 1\right)}$ We want to find a and b such that $ax \leq \arctan x \leq bx$. First, notice that $\lim_{x\to\infty} \dfrac{\arctan x}{x} = 1$, so we conjecture that $b = 1$. To check this, set $f(x) = \arctan x - x$ so that $f'(x) = \frac{1}{1+x^2} - 1 = 0$ only when $x = 0$. This is a local maximum, and since there are no other critical points, we can conclude that $f(x) = \arctan x - x \leq f(0) = 0$ for all $0 \leq x \leq 1$, and hence $\arctan x \leq x$. Since there are points where $\arctan x = x$, 1 must indeed be the maximum value satisfying the condition $\arctan x \leq bx$, and hence $b = 1$. To determine the value of a, we notice that $g(x) = \arctan x$ is concave down when $0 \leq x \leq 1$ because $g''(x) \leq 0$, so every secant line of $g(x)$ lies below $g(x)$. In particular, this includes the secant line connecting the endpoints $(0, \arctan 0) = (0, 0)$ and $(1, \arctan 1) = (1, \frac{\pi}{4})$. This line has slope $\frac{\pi}{4}$, which is our value for a. Therefore, $(a, b) = \left(\frac{\pi}{4}, 1\right)$.

7. [**1088**] For the curve $\sin(x) + \sin(y) = 1$ lying on the first quadrant, find the constant α such that

$$\lim_{x\to 0} x^\alpha \frac{d^2y}{dx^2}$$

exists and is nonzero.

Answer: $\boxed{\frac{3}{2}}$ Differentiate the equation with respect to x to get

$$\cos(x) + \frac{dy}{dx}\cos(y) = 0$$

and again

$$-\sin(x) + \frac{d^2y}{dx^2}\cos(y) - \left(\frac{dy}{dx}\right)^2 \sin(y) = 0.$$

By solving these we have

$$\frac{dy}{dx} = -\frac{\cos(x)}{\cos(y)}$$

and

$$\frac{d^2y}{dx^2} = \frac{\sin(x)\cos^2(y) + \sin(y)\cos^2(x)}{\cos^3(y)}.$$

Let $\sin(x) = t$, then $\sin(y) = 1 - t$. Also $\cos(x) = \sqrt{1-t^2}$ and $\cos(y) = \sqrt{1-(1-t)^2} = \sqrt{t(2-t)}$. Substituting it gives

$$\frac{d^2y}{dx^2} = \frac{t^2(2-t) + (1-t)(1-t^2)}{t^{3/2}(2-t)^{3/2}} = t^{-3/2}\frac{1-t+t^2}{(2-t)^{3/2}}.$$

Since $\lim_{x\to 0} \frac{t}{x} = 1$, $\alpha = \frac{3}{2}$ should give the limit $\lim_{x\to 0} x^\alpha \frac{d^2y}{dx^2} = \frac{1}{2\sqrt{2}}$.

8. [**1152**] Find the volume of the intersection of 3 cylinders that lie in the plane, each of radius 1 and with an angle between each pair of cylindrical axes of $\pi/3$.

   Answer: $\boxed{\frac{28\sqrt{3}}{3}}$ The volume is contained in a regular hexagonal prism which has volume $Ah = 6\sqrt{3} \cdot 2 = 12\sqrt{3}$. The volume of the intersection is this minus the bits on the corners. Consider a cylinder whose axis is parallel to the x-axis. At a height h above the xy-plane, let y be the distance from the cylinder's axis $(\tan\frac{\pi}{6}, 0, h) = (\frac{\sqrt{3}}{3}, 0, h)$ to the point of intersection with an adjacent cylinder $(\frac{\sqrt{3}}{3}, y, h)$, and let ℓ be the distance from $(0,0,h)$ to the base of the hexagon at this height along the x-axis $(\frac{\sqrt{3}}{3}, 0, h)$. We can write:

$$h^2 + x^2 = 1^2 \implies x = \sqrt{1-h^2}$$

   From the perspective of the z-axis, the hexagon lies in the xy-plane. By similar triangles:

$$\frac{1}{\frac{1}{\tan\frac{\pi}{6}}} = \frac{x}{\ell} = \frac{\sqrt{3}}{3}$$

   Hence, integrating outward along the height of the radius there are $(2)(6)(2) = 24$ of the following pieces which can be written individually as:

$$\int_0^1 \frac{\ell x}{2}\,dh = \int_0^1 \frac{x^2\sqrt{3}}{6}\,dh = \int_0^1 \frac{(1-h^2)\sqrt{3}}{6}\,dh = \frac{\sqrt{3}}{6}(2/3) = \frac{\sqrt{3}}{9}$$

   The volume is:

$$12\sqrt{3} - \frac{24\sqrt{3}}{9} = \frac{28\sqrt{3}}{3}$$

9. [**1280**] Three numbers are chosen at random between 0 and 2. What is the probability that the difference between the greatest and least is less than $\frac{1}{4}$?

   Answer: $\boxed{\frac{11}{256}}$ Let X_1, X_2, X_3 be three random variables from the uniform distribution on $[0,2]$. Let $m = \min\{X_1, X_2, X_3\}$ and let $M = \max\{X_1, X_2, X_3\}$. Let $p_m(x)$ be the probability density for the random variable m. In this notation, we are looking for $P(M - m \le \frac{1}{4})$. We can calculate this by conditioning on $m = x$ and integrating. In particular,

$$P(M - m \le \frac{1}{4}) = \int_0^2 p_m(x) P(M \le x + \frac{1}{4} \mid m = x)dx$$

   Where $p_m(x)$ is the probability density for the random variable m at x. We can calculate $p_m(x)$ as follows:

$$p_m(x) = \frac{d}{dx}P(m \le x) = \frac{d}{dx}(1 - P(m \ge x)) = \frac{d}{dx}(1 - P(X_1 \ge x)P(X_2 \ge x)P(X_3 \ge x))$$

$$= \frac{d}{dx}\left(1 - \left(\frac{2-x}{2}\right)^3\right) = \frac{3}{2}\left(\frac{2-x}{2}\right)^2$$

   And it is easy to see that

$$P(M \le x + \frac{1}{4} \mid m = x) = \begin{cases} \left(\frac{\frac{1}{4}}{2-x}\right)^2 & \text{if } x \le \frac{7}{4} \\ 1 & \text{if } x \ge \frac{7}{4} \end{cases}$$

Plugging these two expressions into the integral we are trying to evaluate gives

$$\begin{aligned}\int_0^{\frac{7}{4}} \frac{3}{2}\left(\frac{2-x}{2}\right)^2\left(\frac{\frac{1}{4}}{2-x}\right)^2\,dx + \int_{\frac{7}{4}}^2 \frac{3}{2}\left(\frac{2-x}{2}\right)^2\,dx \\ = \frac{3}{128}\int_0^{\frac{7}{4}}\,dx + \frac{3}{8}\int_{\frac{7}{4}}^2 (2-x)^2\,dx \\ = \frac{21}{512} + \frac{1}{512} = \frac{11}{256}.\end{aligned}$$

Alternate Solution: Call the three numbers x, y, and z. By symmetry, we need only consider the case $2 \geq x \geq y \geq z \geq 0$. Plotted in 3D, the values of (x, y, z) satisfying these inequalities form a triangular pyramid with a leg-2 right isosceles triangle as its base and a height of 2, with a volume of $2 \cdot 2 \cdot \frac{1}{2} \cdot 2 \cdot \frac{1}{3} = \frac{4}{3}$. We now need the volume of the portion of the pyramid satisfying $x - z \leq \frac{1}{4}$. The equation $z = x - \frac{1}{4}$ is a plane which slices off a skew triangular prism along with a small triangular pyramid at one edge of the large triangular pyramid. The prism has a leg-$\frac{1}{4}$ right isosceles triangle as its base and a height of $\frac{7}{4}$, so has volume $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{7}{4} = \frac{7}{2^7}$. The small triangular pyramid also has a leg-$\frac{1}{4}$ right isosceles triangle as its base and a height of $\frac{1}{4}$, so has volume $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{3 \cdot 2^7}$. Then our probability is $\left(\frac{7}{2^7} + \frac{1}{3\cdot 2^7}\right) / \left(\frac{4}{3}\right) = 11/256$.

10. [**1536**] Compute the integral

$$\int_0^\pi \ln(1 - 2a\cos x + a^2)\,dx$$

for $a > 1$.

Answer: $\boxed{2\pi \ln a}$ This integral can be computed using a Riemann sum. Divide the interval of integration $[0, \pi]$ into n parts to get the Riemann sum

$$\frac{\pi}{n}\left[\ln\left(a^2 - 2a\cos\frac{\pi}{n} + 1\right) + \ln\left(a^2 - 2a\cos\frac{2\pi}{n} + 1\right) + \cdots + \ln\left(a^2 - 2a\cos\frac{(n-1)\pi}{n} + 1\right)\right].$$

Recall that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

We can rewrite this sum of logs as a product and factor the inside to get

$$\frac{\pi}{n}\ln\left[\prod_{k=1}^{n-1}\left(a^2 - 2a\cos\frac{k\pi}{n} + 1\right)\right] = \frac{\pi}{n}\ln\left[\prod_{k=1}^{n-1}\left(a - e^{k\pi i/n}\right)\left(a - e^{-k\pi i/n}\right)\right].$$

The terms $e^{\pm k\pi i/n}$ are all of the $2n$-th roots of unity except for ± 1, so the inside product contains all of the factors of $a^{2n} - 1$ except for $a - 1$ and $a + 1$. The Riemann sum is therefore equal to

$$\frac{\pi}{n}\ln\frac{a^{2n} - 1}{a^2 - 1}$$

To compute the value of the desired integral, we compute the limit of the Riemann sum as $n \to \infty$; this is

$$\lim_{n\to\infty}\frac{\pi}{n}\ln\frac{a^{2n} - 1}{a^2 - 1} = \lim_{n\to\infty}\pi\ln\sqrt[n]{\frac{a^{2n} - 1}{a^2 - 1}} = \lim_{n\to\infty}\pi\ln a^2 = \boxed{2\pi\ln a}.$$

Alternate Solution: Let the desired integral be $I(a)$, where we think of this integral as a function of the parameter a. In this solution, we differentiate by a to convert the desired integral to an integral of a rational function in $\cos x$:

$$\frac{d}{da}I(a) = \frac{d}{da}\int_0^\pi \ln(1 - 2a\cos x + a^2)\,dx = \int_0^\pi \frac{2a - 2\cos x}{1 - 2a\cos x + a^2}\,dx.$$

All integrals of this form can be computed using the substitution $t = \tan\frac{x}{2}$. Then $x = 2\arctan t$, so $dx = \frac{2}{1+t^2}\,dt$ and

$$\cos x = \cos(2\arctan t) = 2\cos(\arctan t)^2 - 1 = 2\left(\frac{1}{1+t^2}\right) - 1 = \frac{1-t^2}{1+t^2},$$

so our integral becomes

$$\begin{aligned}\frac{d}{da}I(a) &= \int_0^\infty \frac{2a - 2\frac{1-t^2}{1+t^2}}{1 - 2a\frac{1-t^2}{1+t^2} + a^2}\frac{2}{1+t^2}\,dt = 4\int_0^\infty \frac{a(1+t^2) - (1-t^2)}{(1+t^2) - 2a(1-t^2) + a^2(1+t^2)}\frac{1}{1+t^2}\,dt \\ &= 4\int_0^\infty \frac{(a+1)t^2 + (a-1)}{((a+1)^2t^2 + (a-1)^2)(1+t^2)}\,dt = \frac{2}{a}\int_0^\infty \frac{a^2-1}{(a+1)^2t^2 + (a-1)^2}\,dt + \frac{2}{a}\int_0^\infty \frac{1}{1+t^2}\,dt.\end{aligned}$$

In the first integral, we do the substitution $t = \frac{a-1}{a+1}u$. Then $dt = \frac{a-1}{a+1}du$ and we have

$$= \frac{2}{a}\int_0^\infty \frac{1}{1+u^2}\,du + \frac{2}{a}\int_0^\infty \frac{1}{1+t^2}\,dt = \frac{2}{a}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{2\pi}{a}.$$

Therefore, our desired integral is the integral of the previous quantity, or

$$I = \int_0^\pi \ln(1 - 2a\cos x + a^2)\,dx = 2\pi \ln a.$$

Alternate Solution 2: We can also give a solution based on physics. By symmetry, we can evaluate the integral from 0 to 2π and divide the answer by 2, so

$$\int_0^\pi \ln(1 - 2a\cos x + a^2)dx = \int_0^{2\pi} \ln\sqrt{1 - 2a\cos x + a^2}dx$$

Now let's calculate the 2D gravitational potential of a point mass falling along the x axis towards a unit circle mass centered around the origin. We set the potential at infinity to 0. We also note that, since the 2D gravitational force between two masses is proportional to $\frac{1}{r}$, the potential between two masses is proportional to $-\ln r$. So to calculate the gravitational potential, we integrate $-\ln r$ over the unit circle. But if the point mass is at $(a, 0)$, then the distance between the point mass and the section of the circle at angle x is $\sqrt{1 - 2a\cos x + a^2}$. So we get the integral

$$-\int_0^{2\pi} \ln\sqrt{1 - 2a\cos x + a^2}dx$$

This is exactly the integral we want to calculate! We can also calculate this potential by concentrating the mass of the circle at its center. The circle has mass 2π and its center is distance a from the point mass. So the potential is simply $-2\pi\ln a$. Thus, the final answer is $2\pi\ln(a)$.

Alternate Solution 3: We use Chebyshev polynomials. First, define the Chebyshev polynomial of the first kind to be $T_n(x) = \cos(n \arccos x)$. This is a polynomial in x, and note that $T_n(\cos x) = \cos(nx)$. Note that

$$\begin{aligned}\cos((n+1)x) &= \cos nx\cos x - \sin nx \sin x \\ \cos((n-1)x) &= \cos nx\cos x + \sin nx \sin x,\end{aligned}$$

so that $\cos((n+1)x) = 2\cos nx\cos x - \cos((n-1)x)$ and hence the Chebyshev polynomials satisfy the recurrence $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$. Therefore, the Chebyshev polynomials satisfy the generating function

$$\sum_{n=0}^\infty T_n(x)t^n = \frac{1 - tx}{1 - 2tx + t^2}.$$

Now, substituting $x \mapsto \cos x$ and $t \mapsto a^{-1}$, we have

$$2\sum_{n=0}^\infty \cos(nx)a^{-n+1} = 2\sum_{n=0}^\infty T_n(\cos x)a^{-n+1} = \frac{2a - 2\cos x}{1 - 2a\cos x + a^2}.$$

Then

$$\int_0^\pi \frac{2a - 2\cos x}{1 - 2a\cos x + a^2}\,dx = 2\int_0^\pi \sum_{n=0}^\infty \cos(nx)a^{-n+1}\,dx = 2\sum_{n=0}^\infty \left(a^{-n+1}\int_0^\pi \cos(nx)\,dx\right) = 2\pi a^{-1}.$$

Now, since

$$\ln(1 - 2a\cos x + a^2) = \int \frac{2a - 2\cos x}{1 - 2a\cos x + a^2}\, da,$$

we see that

$$\int_0^\pi \ln(1 - 2a\cos x + a^2)\, dx = \int 2\pi a^{-1}\, da = 2\pi \ln a.$$

Alternate Solution 4: This problem also has a solution which uses the Residue Theorem from complex analysis. It is easy to show that $2\int_0^\pi \ln(1-2a\cos(x)+a^2)dx = \int_0^{2\pi} \ln(1-2a\cos(x)+a^2)dx$. Furthermore, observe that $1 - 2a\cos x + a^2 = (a - e^{ix})(a - e^{-ix})$. Thus, our integral is

$$I = \frac{1}{2}\left(\int_0^{2\pi} \ln[(a-e^{ix})(a-e^{-ix})]dx\right) = \frac{1}{2}\left(\int_0^{2\pi} \ln(a-e^{ix})dx + \int_0^{2\pi} \ln(a-e^{-ix})dx\right),$$

where the integrals are performed on the real parts of the logarithms in the second expression. In the first integral, substitute $z = e^{ix}$, $dz = ie^{ix}dx = izdx$; the resulting contour integral is

$$\oint_{\|z\|=1} \frac{\ln(a-z)}{iz} dz.$$

By the Residue Theorem, this is equal to $2\pi i \operatorname{Res}_{z=0} \frac{\ln(a-z)}{iz} = 2\pi \ln(a)$. The second integral is identical. Thus, the final answer is $\frac{1}{2}(4\pi \ln(a)) = 2\pi \ln(a)$.