

12th Annual Johns Hopkins Math Tournament
Saturday, February 19, 2011

Calculus Subject Test

1. [1025] If $f(x) = (x-1)^4(x-2)^3(x-3)^2$, find $f'''(1) + f''(2) + f'(3)$.
2. [1026] Evaluate the integral

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1 + (\tan x)^{\pi e}}.$$

3. [1028] What is the minimal distance between the curves $y = e^x$ and $y = \ln x$?
4. [1032] Let f be one of the solutions to the differential equation

$$f''(x) - 2xf'(x) - 2f(x) = 0.$$

Supposing that f has Taylor expansion

$$f(x) = 1 + x + ax^2 + bx^3 + cx^4 + dx^5 + \dots$$

near the origin, find (a, b, c, d) .

5. [1040] How many real zeroes does the function $f(x) = \frac{x^{2011}}{2011} + \frac{x^{2010}}{2010} + \dots + x + 1$ have?
6. [1056] Find the maximum value of a and minimum value of b such that $a \leq \frac{\arctan x}{x} \leq b$ for $0 \leq x \leq 1$. Express your answer as an ordered pair (a, b) .
7. [1088] For the curve $\sin(x) + \sin(y) = 1$ lying on the first quadrant, find the constant α such that

$$\lim_{x \rightarrow 0} x^\alpha \frac{d^2y}{dx^2}$$

exists and is nonzero.

8. [1152] Find the volume of the intersection of 3 cylinders that lie in the plane, each of radius 1 and with an angle between each pair of cylindrical axes of $\pi/3$.
9. [1280] Three numbers are chosen at random between 0 and 2. What is the probability that the difference between the greatest and least is less than $\frac{1}{4}$?
10. [1536] Evaluate the integral

$$\int_0^\pi \ln(1 - 2a \cos x + a^2) dx$$

for $a > 1$.