

12<sup>th</sup> Annual Johns Hopkins Math Tournament  
Saturday, February 19, 2011

Algebra Subject Test

1. [1025] For each positive integer  $n$ , let  $i_n$  denote the exponential tower

$$\underbrace{\left( \left( \left( i^i \right)^i \right)^i \right)^i}_{n \text{ times}}$$

where  $i = \sqrt{-1}$ ; for example,  $i_1 = i$ ,  $i_2 = i^i$ , and  $i_3 = (i^i)^i$ . Find  $i_{2011}$ .

**Answer:**  $\boxed{-i}$  Note that

$$\left( \left( \left( i^i \right)^i \right)^i \right)^i = i^{i^{2010}} = i^{-1} = -i.$$

**Alternate Solution:** We work out the first few terms and then try to uncover the general pattern. By definition,  $i_1 = i$ . Next, notice that we can write  $i = e^{i\pi/2}$  and so  $i_2 = i^i = (e^{i\pi/2})^i = e^{-\pi/2}$ . Therefore,  $i_3 = (e^{-\pi/2})^i = e^{-i\pi/2} = -i$ ,  $i_4 = (-i)^i = (e^{-i\pi/2})^i = e^{\pi/2}$ , and  $i_5 = (e^{\pi/2})^i = e^{i\pi/2} = i = i_1$ . Hence we see that the tower repeats in cycles of 4 and in particular, for an integer  $4n + 1$ ,  $i_{4n+1} = i$ . It immediately follows that  $i_{2009} = i$ , and then going two more into the cycle,  $i_{2011} = -i$ .

2. [1026] Consider the curves  $x^2 + y^2 = 1$  and  $2x^2 + 2xy + y^2 - 2x - 2y = 0$ . These curves intersect at two points, one of which is  $(1, 0)$ . Find the other one.

**Answer:**  $\boxed{\left(-\frac{3}{5}, \frac{4}{5}\right)}$  From the first equation, we get that  $y^2 = 1 - x^2$ . Plugging this into the second one, we are left with

$$\begin{aligned} 2x^2 - 2x\sqrt{1-x^2} + 1 - x^2 - 2x + 2\sqrt{1-x^2} &= 0 \Rightarrow x^2 - 2x + 1 = (x-1)^2 = 2\sqrt{1-x^2}(x-1) \\ &\Rightarrow x-1 = 2\sqrt{1-x^2} \text{ assuming } x \neq 1 \\ &\Rightarrow x^2 - 2x + 1 = 4 - 4x^2 \Rightarrow 5x^2 - 2x - 3 = 0. \end{aligned}$$

The quadratic formula yields that  $x = \frac{2 \pm 8}{10} = 1, -\frac{3}{5}$  (we said that  $x \neq 1$  above but we see that it is still valid). If  $x = 1$ , the first equation forces  $y = 0$  and we easily see that this solves the second equation. If  $x = -\frac{3}{5}$ , then  $y = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$ . Hence the other point is  $(-\frac{3}{5}, \frac{4}{5})$ .

**Remark:** A shorter solution, albeit one requiring knowledge of abstract algebra is presented: to find the solutions, we compute a Gröbner basis for the ideal  $I = (2x^2 + 2xy + y^2 - 2x - 2y, x^2 + y^2 - 1) \subset \mathbb{R}[x, y]$  using the lexicographic monomial order  $x > y$  to eliminate  $x$ , obtaining  $g_1 = 2x + y^2 + 5y^3 - 2$  and  $g_2 = 5y^4 - 4y^3$ . Hence  $5y^4 = 4y^3$  and  $y = 0$  or  $y = \frac{4}{5}$ . Substituting these values into  $g_1 = 0$  and solving for  $x$  we find the two intersection points are the same as above.

3. [1028] Let  $F(x)$  be a real-valued function defined for all real  $x \neq 0, 1$  such that

$$F(x) + F\left(\frac{x-1}{x}\right) = 1 + x.$$

Find  $F(2)$ .

**Answer:**  $\boxed{\frac{3}{4}}$  Setting  $x = 2$ , we find that  $F(2) + F\left(\frac{1}{2}\right) = 3$ . Now take  $x = \frac{1}{2}$ , to get that  $F\left(\frac{1}{2}\right) + F(-1) = \frac{3}{2}$ . Finally, setting  $x = -1$ , we get that  $F(-1) + F(2) = 0$ . Then we find that

$$\begin{aligned} F(2) &= 3 - F\left(\frac{1}{2}\right) = 3 - \left(\frac{3}{2} - F(-1)\right) = \frac{3}{2} + F(-1) = \frac{3}{2} - F(2) \\ &\Rightarrow F(2) = \frac{3}{4}. \end{aligned}$$

**Alternate Solution:** We can explicitly solve for  $F(x)$  and then plug in  $x = 2$ . Notice that for  $x \neq 0, 1$ ,  $F(x) + F\left(\frac{x-1}{x}\right) = 1 + x$  so

$$F\left(\frac{x-1}{x}\right) + F\left(\frac{1}{1-x}\right) = 1 + \frac{x-1}{x} \text{ and } F\left(\frac{1}{1-x}\right) + F(x) = 1 + \frac{1}{1-x}.$$

Thus

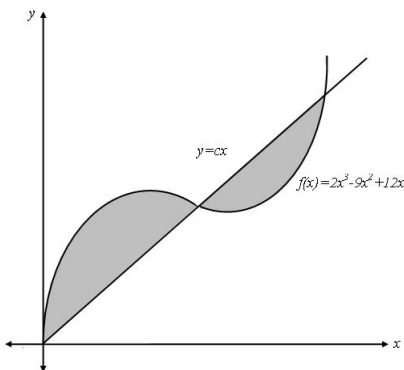
$$\begin{aligned} 2F(x) &= F(x) + F\left(\frac{x-1}{x}\right) - F\left(\frac{x-1}{x}\right) - F\left(\frac{1}{1-x}\right) + F\left(\frac{1}{1-x}\right) + F(x) \\ &= 1 + x - \left(1 + \frac{x-1}{x}\right) + 1 + \frac{1}{1-x} \\ &= 1 + x + \frac{1-x}{x} + \frac{1}{1-x}. \end{aligned}$$

It follows that  $F(x) = \frac{1}{2} \left(1 + x + \frac{1-x}{x} + \frac{1}{1-x}\right)$  and the result follows by taking  $x = 2$ .

4. [1032] Let  $p$  be a monic cubic polynomial such that the sum of the coefficients, the sum of the roots, and the sum of each root squared are all equal to 1. Find  $p$ . Note: monic means that the leading coefficient of  $p$  is 1.

**Answer:**  $x^3 - x^2 + 1$  Let  $p(x) = x^3 + ax^2 + bx + c$ . By Vieta's Formulas, the sum of all the roots of  $p$  is  $-\frac{a}{1} = -a$ , which we are told is 1, so  $a = -1$ . Since the sum of the coefficients is 1,  $1 + a + b + c = 1$ , and since  $a = -1$ ,  $b + c = 1$ . If  $S_i = r_1^i + r_2^i + r_3^i$  (where  $r_j$  are the roots), then  $S_2 + aS_1 + 2b = 0$  (this can be proved using symmetric polynomials). As we are given  $S_1 = S_2 = 1$ , and we already determined  $a = -1$ ,  $1 - 1 + 2b = 0$ , so  $b = 0$ . Finally, as  $b + c = 1$  from before, it follows that  $c = 1$  and so  $p(x) = x^3 - x^2 + 1$ .

5. [1040] The line  $y = cx$  is drawn such that it intersects the curve  $f(x) = 2x^3 - 9x^2 + 12x$  at two points in the first quadrant, creating the two shaded regions as shown in the diagram (not to scale). If the areas of the two shaded regions are the same, what is  $c$ ?



**Answer:** 3 Every cubic polynomial  $f$  is point symmetric, meaning there exists a point such that  $f$  is antisymmetric about that point. By translating  $f$  to the origin about this point,  $f$  becomes an odd function. Thus we simply need to determine this point. Let  $g$  be  $f$  translated to the origin (so  $g$  is an odd function). Then  $g$  takes the form  $g(x) = ax(x-c)(x+c) = ax(x^2 - c^2)$  because  $x = 0$  is one root, and the other two roots are additive inverses of each other,  $\pm c$  (the  $a$  term is simply the coefficient on the highest order term). To shift  $f$  to the origin, we need a horizontal shift and a vertical shift. That is, there exists  $(b, d)$  such that  $a(x-b)((x-b)^2 - c^2) + d = ax(x^2 - c^2)$ . The point  $(b, d)$  is the point around which  $f$  is symmetric. In the problem,  $f(x) = 2x^3 - 9x^2 + 12x$ , so we set  $2x^3 - 9x^2 + 12x = a(x-b)((x-b)^2 - c^2) + d$ . This gives a system of (nonlinear) equations:

$$a = 2 \quad -3ab = -9 \quad 3ab^2 - ac^2 = 12 \quad -ab(b^2 - c^2) + d = 0.$$

Solving gives  $(b, d) = (1.5, 4.5)$ . Now, the function  $g(x) := f(x + 1.5) - 4.5$  is odd. This means that for any line  $\ell(x)$  through the origin, if  $\pm a$  are the  $x$ -coordinates of the points of intersection of  $\ell(x)$

and  $g(x)$ , the areas between  $-a$  and  $0$ , and  $0$  and  $a$  bounded by the two functions are the same. If we translate  $g(x)$  back to  $f(x)$ , then the areas still have to agree and the only line  $\ell(x) = cx$  passing through  $(1.5, 4.5)$  is  $\ell(x) = 3x$ .

**Alternate (Calculus) Solution:** As we see, the line  $cx$  intersects the curve at two points in the first quadrant, say  $\alpha$  and  $\beta$  with  $\alpha < \beta$ . Since the areas of the shaded regions must be equal, this means

$$\int_0^\alpha 2x^3 - 9x^2 + 12x - cx \, dx = \int_\alpha^\beta cx - 2x^3 + 9x^2 - 12x \, dx.$$

If  $F(x)$  denotes the antiderivative of  $f(x)$ , then the above gives that

$$F(\alpha) - \frac{c\alpha^2}{2} = \frac{c\beta^2}{2} - F(\beta) - \frac{c\alpha^2}{2} + F(\alpha)$$

since  $F(0) = 0$ . Rearranging,

$$F(\beta) = \frac{\beta^4}{2} - 3\beta^3 + 6\beta^2 = \frac{c\beta^2}{2} \Rightarrow \beta^2 - 6\beta + 12 = c,$$

where division by  $\beta$  is allowed because  $\beta > 0$ . However, going back to our original function, by definition  $\beta$  is a solution to  $f(\beta) = c\beta$ , or  $2\beta^2 - 9\beta + 12 = c$ . Setting this expression equal to the one above for  $c$ , setting a common denominator, and combining terms gives that  $\beta = 3$ . Finally, because  $2\beta^2 - 9\beta + 12 = c$ , using  $\beta = 3$  shows that  $c = 3$ .

6. [1056] Let  $p(x) = (x^3 + x + 1)^{2011}$ . Let  $\omega = e^{2\pi i/5}$ . Compute  $p(\omega)p(\omega^2)p(\omega^3)p(\omega^4)$ .

**Answer:** 1 Note that as  $\omega$  is a fifth root of unity, all of  $\omega, \omega^2, \omega^3, \omega^4$  are roots of  $t^4 + t^3 + t^2 + t + 1 = 0$ . Then  $t^3 + t + 1 = -t^4 - t^2$ . Therefore, for  $x = \omega, \omega^2, \omega^3, \omega^4$ ,

$$p(x) = (-x^4 - x^2)^{2011} = -x^{4022}(x^2 + 1)^{2011} = x^{4022}(i + x)^{2011}(i - x)^{2011}.$$

Finally, one has  $A^4 + A^3 + A^2 + A + 1 = (1 - \omega)(1 - \omega^2)(1 - \omega^3)(1 - \omega^4)$  as a property of roots of unity, so

$$p(\omega)p(\omega^2)p(\omega^3)p(\omega^4) = (\omega^{1+2+3+4})^{4022}(i^4 + i^3 + i^2 + i + 1)^{2011}(i^4 - i^3 + i^2 - i + 1)^{2011} = 1.$$

7. [1088] Find all integers  $x$  for which  $|x^3 + 6x^2 + 2x - 6|$  is prime.

**Answer:** 1, -1 The whole equation is  $\equiv 0 \pmod{3}$ , so  $x^3 + 6x^2 + 2x - 6$  should be  $3$  or  $-3$ . The equation  $(x^3 + 6x^2 + 2x - 6)^2 = 3^2$  is same as  $(x - 1)(x^2 + 7x - 9)(x + 1)(x^2 + 5x - 3) = 0$ , so only  $1$  and  $-1$  work for  $x$ .

8. [1152] Find the final non-zero digit in  $100!$ . For example, the final non-zero digit of  $7200$  is  $2$ .

**Answer:** 4 We first claim that  $100!$  ends in  $24$  zeroes. Indeed, it suffices to count the number of  $5$ 's in the prime factorization of  $100!$ . There are  $20$  multiples of  $5$  up to  $100$ , which gives  $20$  zeroes, and then  $25, 50, 75,$  and  $100$  each contribute one more for a total of  $24$ . Now, let  $p(k)$  denote the product of the first  $k$  positive multiples of  $5$ , and notice that  $p(k) = 5^k \cdot k!$ . Also, by cancelling terms of  $p(k)$ , we have that  $\frac{(5k)!}{p(k)} \equiv (1 \cdot 2 \cdot 3 \cdot 4)^k \equiv (-1)^k \pmod{5}$ . From our claim, we can write  $100! = M \cdot 10^{24}$ , where

$$M = 2^{-24} \cdot \frac{100!}{p(20)} \cdot \frac{20!}{p} (4) \cdot 4! \equiv (2^4)^{-6} \cdot (-1)^{20} \cdot (-1)^4 \cdot (-1) \equiv -1 \pmod{5}.$$

Since more  $2$ 's than  $5$ 's divide  $100!$ , the last nonzero digit must be even, and so it is  $4$ .

9. [1280] Let  $T_n$  denote the number of terms in  $(x + y + z)^n$  when simplified, i.e. expanded and like terms collected, for non-negative integers  $n \geq 0$ . Find

$$\sum_{k=0}^{2010} (-1)^k T_k = T_0 - T_1 + T_2 - \cdots - T_{2009} + T_{2010}$$

**Answer:**  $\boxed{1006^2}$  First note that the expression  $(x + y + z)^n$  is equal to

$$\sum \frac{n!}{a!b!c!} x^a y^b z^c$$

where the sum is taken over all non-negative integers  $a$ ,  $b$ , and  $c$  with  $a + b + c = n$ . The number of non-negative integer solutions to  $a + b + c = n$  is  $\binom{n+2}{2}$ , so  $T_k = \binom{k+2}{2}$  for  $k \geq 0$ . It is easy to see that  $T_k = 1 + 2 + \cdots + (k + 1)$ , so  $T_k$  is the  $(k + 1)$ st triangular number. If  $k = 2n - 1$  is odd and we let  $t_i$  denote the  $i$ th triangular number, then

$$\sum_{j=1}^k (-1)^{j+1} t_j = n^2.$$

(For a quick visual proof of this fact, we refer the reader to <http://www.jstor.org/stable/2690575>.) Therefore, since  $T_{2010}$  is the 2011th triangular number and  $2011 = 2(1006) - 1$ , we can conclude that the desired sum is  $1006^2$ .

10. [1536] Define a sequence  $(a_n)$  by

$$\begin{aligned} a_0 &= 1 \\ a_1 &= a_2 = \cdots = a_7 = 0 \\ a_n &= \frac{a_{n-8} + a_{n-7}}{2} \text{ for } n \geq 8 \end{aligned}$$

Find the limit of this sequence.

**Answer:**  $\boxed{\frac{1}{15}}$  Define a sequence  $(b_n)$  by

$$b_n = a_n + 2a_{n+1} + 2a_{n+2} + \cdots + 2a_{n+7}.$$

Now, we observe that

$$\begin{aligned} b_{n+1} &= a_{n+1} + 2a_{n+2} + 2a_{n+3} + \cdots + 2a_{n+8} \\ &= a_{n+1} + 2a_{n+2} + \cdots + 2a_{n+7} + (a_n + a_{n+1}) \\ &= a_n + 2a_{n+1} + 2a_{n+2} + \cdots + 2a_{n+7} \\ &= b_n. \end{aligned}$$

However, if  $\ell$  denotes the limiting value of the sequence  $(a_n)$ , then we have that  $b_0$  is equal to the limiting value of  $(b_n)$ , which must be  $(2(8) - 1) \cdot \ell = 15\ell$ . However, we can compute  $b_0 = 1$ , so the limiting value of the sequence  $(a_n)$  is  $\frac{1}{15}$ .