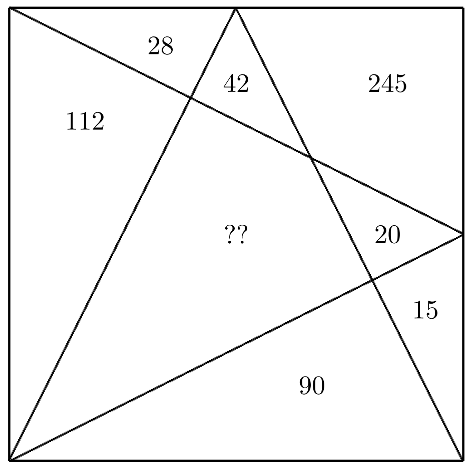
11^{th} Annual Johns Hopkins Math Tournament
Sunday, April 11, 2010
Grab Bag-Lower Division: Solutions

(1) (**6**) Below is a square, divided by several lines (not to scale). Several regions have their areas written inside. Find the area of the remaining region.



Answer: 288

Solution: Each large triangle has a base and height equal to the side length of the square. Hence each triangle covers half of the total area. Thus the area that is double-covered is equal to the area that is not covered. Hence the area is $245 + 15 + 28 = 288$.

(2) (**7**) Three unit circles ω_1, ω_2, and ω_3 in the plane have the property that each circle passes through the centers of the other two. Find the area of the region that is intersected by all of ω_1, ω_2, and ω_3.

Answer: $\frac{\pi-\sqrt{3}}{2}$

Solution: Let P, Q, and R denote the respective centers of ω_1, ω_2, and ω_3, and let Ω denote the common intersection. Since ω_1 passes through Q and R, $PQ = PR = 1$. Similarly, we can find that $QR = 1$. Therefore, $\triangle PQR$ is an equilateral triangle contained within Ω. In particular, Ω is spanned by arcs PQ, PR, and QR. Since $\triangle PQR$ is equilateral, the area of the sector spanned by each of these arcs is $\frac{\pi}{6}$. The area of Ω is the sum of the areas of each of the arcs, minus the regions double-counted. Equilateral triangle $\triangle PQR$ is double-counted twice, and has area $\frac{\sqrt{3}}{4}$. Hence the area of $\Omega = 3 \cdot \frac{\pi}{6} - 2 \cdot \frac{\sqrt{3}}{4} = \frac{\pi-\sqrt{3}}{2}$.

(3) (**8**) Two integers are called *relatively prime* if they share no common prime factors (that is, their greatest common divisor is 1). Given that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$, find the probability that two integers picked at random are relatively prime.

Answer: $\frac{6}{\pi^2}$

Solution: The probability that two integers both have a given prime p as a factor is $\frac{1}{p^2}$. Then the probability that they don't have p as a common factor is $1 - \frac{1}{p^2}$. Therefore, the probability that two numbers have no common prime factors is

$$P = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\left(1 - \frac{1}{11^2}\right)\cdots.$$

Using the fact that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \text{ when } |x| < 1,$$

we can write this as

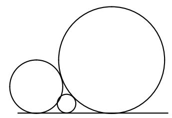
$$P = \left(\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \cdots\right)\left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \cdots\right)\cdots\right)^{-1}.$$

Since every positive integer can be expressed as a product of primes in a unique way, the previous expression is equivalent to

$$P = \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right)^{-1} = \frac{6}{\pi^2}.$$

Remark: The alert reader might immediately recognize that the first expression for P given above is equal to $\frac{6}{\pi^2}$ by Euler's amazing formula. We refer the interested reader to `http://en.wikipedia.org/wiki/Proof_of_the_Euler_product_formula_for_the_Riemann_zeta_function`.

(4) (**10**) Two circles with radii 1 and 2 are arranged in such a way that they are tangent to each other and a straight line. A third circle is placed between them so that it is tangent to both circles and the line, as shown in the diagram below (not drawn to scale). Find the radius of the small circle.



Answer: $6-4\sqrt{2}$

Solution: Let the circle of radius 1 have center A, the circle of radius 2 have center B, and the third circle have center C. Let DE be the horizontal line passing through C, where D lies on the radius of the circle of radius 1 perpendicular to the tangent line, and E lies on the radius of the circle of radius 2 perpendicular to the tangent line. Let $DC = a$, $CE = b$, and the radius of the third circle have radius r. Applying the Pythagorean Theorem to $\triangle ADC$ and $\triangle BCD$,

$$(1-r)^2 + a^2 = (1+r)^2 \text{ and } (2-r)^2 + b^2 = (2+r)^2.$$

Therefore, $a^2 = 4r$ and $b^2 = 8r$. Since $(a+b)^2 + 1 = 9$, $2\sqrt{2} + 2\sqrt{2r} = \sqrt{8}$. It follows that $r = 6 - 4\sqrt{2}$.

(5) (**11**) A box contains 4 green balls, 4 blue balls, 2 red balls, a yellow ball, a white ball, and a black ball. The balls are picked randomly, one at a time without replacement, until two balls of the same color have been removed. The process requires that at most 7 balls be removed. Find the probability that 7 balls are removed. You may leave your answer in terms of $\binom{n}{k}$ or as an explicit probability.

Answer: $\frac{32}{\binom{13}{6}} = \frac{8}{429}$

Solution: The desired probability is the same as the probability that upon drawing the first 6 balls, no two are of the same color. But this is possible if and only if those 6 balls each have a different color, and there are $4 \cdot 4 \cdot 2 \cdot 1 \cdot 1 \cdot 1 = 32$ possibilities out of $\binom{13}{6}$ possible draws. So the desired probability is

$$\frac{32}{\binom{13}{6}} = \frac{32}{1716} = \frac{8}{429}.$$

(6) (**13**) Jason, Ben, Yichen, and Anthony are playing Contract Bridge. After one hand, they notice that all of the cards of two suits are split between Jason and Ben. Determine the number of ways 13 cards can be dealt to each player such that this is the case. You may leave your answer in $\binom{n}{k}$ form. (Contract Bridge is played with a standard 52-card deck.)

Answer: $\binom{4}{2}\binom{26}{13}^2 = 6 \cdot \binom{26}{13}^2$

Solution: If two complete suits are in the union of Jason's and Ben's hands, then the other two complete suits are in Yichen's and Anthony's hands. There are $\binom{4}{2} = 6$ ways that the suits can be distributed in this way. Since for each pair, some player has some 13 of the 26 possible cards, there are $\binom{4}{2}\binom{26}{13}^2 = 6 \cdot \binom{26}{13}^2$ possible deals.

(7) (**14**) Fifteen chairs are lined up in a row for Professor Zucker's Honors Linear Algebra Exam. However, only 6 students show up and Zucker won't let any two students sit next to each other. In how many ways can Zucker arrange his students?

Answer: $151,200$

Solution: Let t denote a taken seat, and e denote an empty seat. So, for example, *eteteeteteteete* is a possible seating arrangement. Now, if we remove and empty seat from between each pair of neighboring taken seats (*ettetttete* in the sequence above), then we have a sequence without any restrictions. In similar spirit, we can go from a situation with 4 e's and 6 t's with no restrictions to one as in the problem by adding an e between each pair of neighboring t's. Therefore, we have created a one-to-one correspondence between the problem at hand and seating 6 people in $15-5 = 10$ seats without restrictions. Thus there are $\binom{10}{6} = 210$ ways to select the students and $6! = 720$ ways to arrange the students in the seats, for a total of $(720)(210) = 151,200$ seatings.

(8) (**15**) A point P is randomly placed on $\overline{AB}$. Let M be the midpoint of $\overline{AB}$. What is the probability that that AP, PB, and AM can be made to form a triangle? That is, what is the probability that AP, PB, and AM are possible lengths for a triangle?

Answer: $\frac{1}{2}$

Solution: Without loss of generality, let $AB = 2$, so that $AM = 1$. Then either $AP > AM$ or $PB > AM$. Suppose that AP is the bigger side. Then we need $PB + AM > AP$ if the three lengths are to form a triangle. Let $PB = x$; since $PB \le AP$, $x \le 1$. Then we must have that $2 - x < x + 1$, from which it follows that $\frac{1}{2} < x$ in order for the three lengths to form a triangle. Since $0 \le x \le 1$, the probability is $\frac{1}{2}$.

(9) (**16**) Let $\triangle ABC$, be acute with perimeter 100. Let D be a point on $\overline{BC}$. The circumcircles of ABD and ADC intersect $\overline{AC}$ and $\overline{AB}$ at E and F respectively such that $DE = 14$ and $DF = 11$. If $\angle EBC \cong \angle BCF$, find $\frac{AE}{AF}$.

Answer: $\frac{74}{101}$

Solution: Since $BDEA$ is cyclic, $\angle EBD \cong EAD$. Similarly, $\angle DCF \cong \angle DAF$. Since $\angle EBC \cong \angle BCF$ by assumption, $\angle DAB \cong \angle CAB$. Since CD and DF are intersected by congruent angles in the same circle, $DF = CD = 11$. Similarly, $DB = 14$. By the angle bisector theorem, $AC = 11x$ and $AB = 14x$. Since the perimeter of $\triangle ABC = 100$, $25 + 25x = 100$, meaning $x = 3$. So $AC = 33$ and $AB = 42$. By the power of a point theorem from B, $BF = \frac{BD \cdot BC}{BA} = \frac{14 \cdot 25}{42} = \frac{25}{3}$ and $CE = \frac{11 \cdot 25}{33} = \frac{25}{3}$. Subtracting these lengths from AB and AC respectively, we see that $AF = 42 - \frac{25}{3} = \frac{101}{3}$ and $AE = 33 - \frac{25}{3} = \frac{74}{3}$. It follows that the answer is $\frac{AE}{AF} = \frac{74}{101}$.