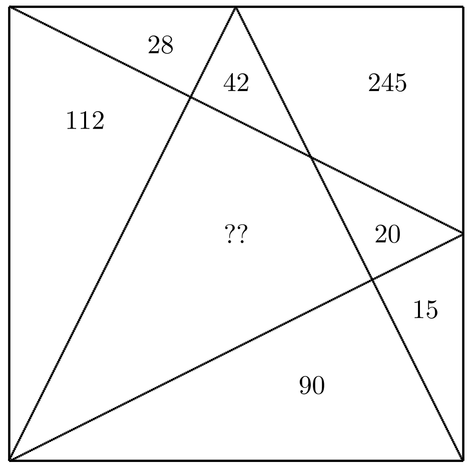
11^{th} Annual Johns Hopkins Math Tournament
Sunday, April 11, 2010
Grab Bag-Upper Division: Solutions

(1) (**7**) Below is a square, divided by several lines (not to scale). Several regions have their areas written inside. Find the area of the remaining region.



Answer: 288
Solution: Each large triangle has a base and height equal to the side length of the square. Hence each triangle covers half of the total area. Thus the area that is double-covered is equal to the area that is not covered. Hence the area is $245 + 15 + 28 = 288$.

(2) (**8**) A line is drawn tangent to the graph of $f(x) = \frac{1}{x}$ at the point $(a, f(a))$ in the first quadrant. The tangent line, x- and y-axes form a triangle. Find the area of the triangle in terms of a.
Answer: $\frac{1}{2}$
Solution: We compute the equation of the tangent line passing through the point $(a, \frac{1}{a})$: $f'(x) = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ so the slope of the line passing through the point is $-\frac{1}{a^2}$. Therefore, since a line has equation $y - y_0 = m(x - x_0)$ where m is the slope, we have that

$$y - \frac{1}{a} = -\frac{1}{a^2}(x - a) \Longrightarrow y = -\frac{x}{a^2} + \frac{2}{a}.$$

Now we just need to compute the base and height of the triangle, which we can do by solving for the x- and y-intercepts. Setting $x = 0$, we find that the y-intercept is $\frac{2}{a}$ and setting $y = 0$, we find that the x-intercept is $\frac{a}{2}$. Therefore, the area of the triangle is $\frac{1}{2} \cdot \frac{2}{a} \cdot \frac{a}{2} = \frac{1}{2}$. In particular, the area does not depend on a.

(3) (**10**) Let $x > 0$. If $\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$, find $\Gamma\left(\frac{1}{2}\right)$ where $\Gamma(x)$ is the function defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt$$

Answer: $\sqrt{\pi}$
Solution: We actually show something more general and deduce our problem as a special case. Let $u = t^x$, $du = xt^{x-t}\,dt$. Then

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt = \int_0^\infty e^{-u^{1/x}} \frac{du}{x} = \frac{1}{x}\int_0^\infty e^{-u^{1/x}}\,du$$

Therefore, if we substitute $x = \frac{1}{2}$ and use the fact that $\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$, we find that

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}\,du = \sqrt{\pi}$$

Remark: This function is called the Gamma function and is an extremely important function in many branches of math. The Gamma function enjoys many interesting properties. Perhaps the simplest is that it is an extension of the factorial function. We refer the interested reader to `http://en.wikipedia.org/wiki/Gamma_function`.

(4) (**12**) Fifteen chairs are lined up in a row for Professor Zucker's Honors Linear Algebra Exam. However, only 6 students show up and Zucker won't let any two students sit next to each other. In how many ways can Zucker arrange his students?
Answer: $151,200$
Solution: Let t denote a taken seat, and e denote an empty seat. So, for example, *eteteeteteteete* is a possible seating arrangement. Now, if we remove and empty seat from between each pair of neighboring taken seats (*ettetttete* in the sequence above), then we have a sequence without any restrictions. In similar spirit, we can go from a situation with 4 e's and 6 t's with no restrictions to one as in the problem by adding an e between each pair of neighboring t's. Therefore, we have created a one-to-one correspondence between the problem at hand and seating 6 people in 15-5=10 seats without restrictions. Thus there are $\binom{10}{6} = 210$ ways to select the students and $6! = 720$ ways to arrange the students in the seats, for a total of $(720)(210) = 151,200$ seatings.

(5) (**13**) Evaluate the following limit:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}$$

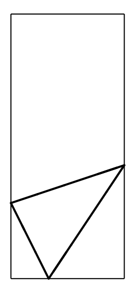
Answer: $\ln 2$
Solution: Writing out the expression, the desired limit is

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)$$

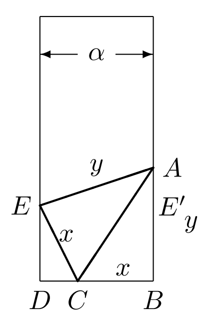
In particular, notice that this is just the Riemann sum for $\frac{1}{x+1}$ on the interval $[0,1]$. Therefore, the desired limit is simply

$$\int_0^1\frac{dx}{x+1} = \ln(x+1)\Big|_0^1 = \ln 2$$

(6) (**15**) The lower right-hand corner of a page is folded over so that it just touches the left edge of the paper, as shown in the figure below. If the width of the paper is α and the page is very long, find the minimum length of the crease if the lower left corner is held fixed.



Answer: $\frac{3\sqrt{3}\alpha}{4}$
Solution: If $x = \overline{BC}$ and $y = \overline{AB}$,



then we have

$$ED = \sqrt{x^2-(\alpha-x)^2} = \sqrt{2\alpha x-\alpha^2} \text{ from } \triangle EDC$$
$$\alpha^2+(y-\overline{ED})^2 = y^2 \qquad \text{from } \triangle EE'A,$$

so

$$\alpha^2+\left(y-\sqrt{2\alpha x-\alpha^2}\right)^2 = y^2$$
$$-y\sqrt{2\alpha x\alpha^2}+\alpha x = 0$$
$$y^2\left(2\alpha x-\alpha^2\right) = \alpha^2x^2$$

$$y^2 = \frac{\alpha^2 x^2}{2\alpha x - \alpha^2} = \frac{\alpha x^2}{2x - \alpha}.$$

The square of the length of the crease is

$$x^2 + y^2 = x^2 + \frac{\alpha x^2}{2x - \alpha} = \frac{2x^3}{2x - \alpha},$$

so the length is smallest when

$$0 = 6x^2(2x - \alpha) - 4x^3 = 8x^3 - 6x^2\alpha = x^2(8x - 6\alpha),$$

or $x = \frac{3\alpha}{4}$. For this x, the length is

$$\sqrt{\frac{2\left(\frac{3\alpha}{4}\right)^3}{\frac{3\alpha}{2} - \alpha}} = \frac{3\sqrt{3}\alpha}{4}.$$

(7) (**17**) Let $f(x) = x^6 - 3x^2 + x$. The graph of f has three real critical points. Find the unique quadratic equation which passes through these three points.
Answer: $-2x^2 + \frac{5}{6}x$
Solution: The critical points of this polynomial occur at points $x = c_i$ such that $f'(x) = 0 = 6x^5 - 6x + 1$. But the values of the parabola, $g(x)$, must be the same as those of the function at c_i. Hence we can let $g(x) = f(x) - \alpha(x)f'(x)$ for any function α. Now, choose $\alpha(x) = \frac{x}{6}$:

$$g(x) = x^6 - 3x^2 + x - \frac{x}{6}\left(6x^5 - 6x + 1\right) = -2x^2 + \frac{5}{6}x.$$

Becasue g has the same value as f when $x = c_1, c_2, c_3$ and a parabola is defined by 3 points, this is the unique parabola that we want.

(8) (**18**) Evaluate the integral $\displaystyle\int_0^1 (e-1)\sqrt{\ln(1 + ex - x)} + e^{x^2}\, dx$.
Answer: e
Solution: Let A denote the region above the graph of $y = e^{x^2}$ and B the region below the graph (within the box $[0,1] \times [0,e]$). Then area(A) + area$(B) = e$ since the union of A and B is the box $[0,1] \times [0,e]$. It's easy to check that $x = \sqrt{\ln y}$ is the inverse of $y = e^{x^2}$. Therefore, area$(A) = \int_1^e \sqrt{\ln y}\, dy$ (substituting $x = 0$ and $x = 1$ for the limits of integration). By definition, area$(B) = \int_0^1 e^{x^2}\, dx$. To get the desired integral, we must change the limits of integration in the first integral from 1 and e to 0 and 1, which can be done under the substitution $x = \frac{y-1}{e-1}$. This gives $ex - x + 1 = y$, $(e-1)\, dx = dy$ and so area$(A) = \int_0^1 (e-1)\sqrt{\ln(1 + ex - x)}\, dx$. Therefore the original integral is simply the area(A) + area(B), which we saw to be e.