11^{th} Annual Johns Hopkins Math Tournament
Sunday, April 11, 2010

Explorations Unlimited Round-Introduction to Group Theory: Solutions

Q1) Inverse: $-q$ [1 point]
Identity: 0 [1 point]

Q2) $\mathbb{Z} \setminus \{0\}$ does NOT form a group under addition [1 point]
± 1 are the only elements with an inverse/no numbers other than ± 1 has an inverse [2 points]

Q3) $\mathbb{Q}$ IS an abelian group under addition [1 point]
The center is $\mathbb{Q}$ [1 point]
$\mathbb{R} \setminus \{0\}$ IS an abelian group under multiplication [1 point]
The center is $\mathbb{R} \setminus \{0\}$ [1 point]

Q4) $\{1, 2, 3\}$, $\{1, 3, 2\}$, $\{2, 1, 3\}$, $\{2, 3, 1\}$, $\{3, 1, 2\}$, $\{3, 2, 1\}$ [1 point each]

Q5) $(1\,5)(2\,7\,3)(4)(6)$ [1 point for each correct cycle]
$1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 4$, $4 \mapsto 2$ [1 point if 2 correct, 2 points if 3 correct, 3 points if all correct]

Q6) $(1\,2\,7\,5\,6\,4\,3)$ [1 point for each correct number, 0 points if does not start with 1]

Q7) $(3\,4\,6\,5\,7\,2\,1)$ (if Q6 is wrong, their answer should reverse the numbers from their Q6 answer) [1 point for each correct number]

Q8) $(1\,4\,3)(2)$ [2 points if completely correct, 0 points otherwise]
$(1\,2\,3)(4)$ [2 points if completely correct, 0 points otherwise]
S_4 is NOT abelian [2 points]

Q9) An m-cyle has m possible elements $(- \; - \cdots -)$. The first space can be anything from $\{1, \ldots, n\}$ (since we're in S_n) so there are n possibilities. The second space can be anything other than what was in the first spot, so there are $n-1$ possibilities. We can continue on to the mth spot, where we can choose anything but the previous $m-1$ choices. Hence there are $n-m+1$ possibilities. Therefore, there are $n \cdot n-1 \cdots n-m+1$ ways to form an m-cycle. However, two m-cycles are the same if we simply rotate them. Since there are m possible rotations, there are m possible representations, and so there are $\frac{n \cdot n-1 \cdots n-m+1}{m}$ possible m-cycles in S_n [7 points if the numerator is correct, 3 points if the denominator is correct]

Q10) $\{i, (1\,2\,3), (1\,3\,2)\}$, $\{i, (1\,2)\}$, $\{i, (2\,3)\}$, or $\{i, (1\,3)\}$ are all acceptable answers [4 points if completely correct, 0 points otherwise]

Q11) $4\mathbb{Z}$ is not empty [1 point if they say it's not empty]
$4x - 4y = 4(x-y)$ is in $4\mathbb{Z}$ [3 points]
It IS a subgroup [1 point]

Q12) Order is 3 [4 points]

Q13) Order is 2 [2 points]

Q14) $(1)(2)(3)$, $(1\,2\,3)$, $(1\,3\,2)$ [1 point for each completely correct answer]

Q15) Since H is a subgroup and subgroups are closed, if a is already in H, then $a \circ h$ is still in H. So $aH = H$/they're equal/the same [5 points]

Q16) $(aH)_{(n)} = (aH) \circ (aH) \circ \cdots \circ (aH) = (a \circ \cdots a)H = a_{(n)}H$ [7 points]

Q17) If h is in H, the function is $f(h) = ah$ [4 points]
If $ah_1 = ah_2$, divide by a on both sides to get $h_1 = h_2$ [3 points]
Bijections preserve order so the order of H must be the same as the order of aH [4 points]
$k = \frac{|G|}{n} = \frac{|G|}{|H|}$, so $|H| = \frac{|G|}{k}$ [4 points]

Q18) By the hint, $|x| = |\langle x\rangle|$ so applying Lagrange's Theorem to $H = \langle x\rangle$, the claim follows [5 points]
From the first part, $|G|$ is a multiple of the order of x. If k is the order of x and $|G| = km$, then $x_{(|G|)} = x_{(km)} = 1_{(m)} = 1$ [5 points]

Q19) Let x be in G be anything but the identity. Then $|\langle x\rangle| > 1$ and $|\langle x\rangle|$ divides $|G|$. Since $|G|$ is prime and $|\langle x\rangle| > 1$, $|\langle x\rangle| = p$. Hence $G = \langle x\rangle$, which means G is cyclic [10 points]

Q20) Because f is a homomorphism, $f(a_{(n)}) = f(a \circ \cdots \circ a) = f(a) \cdots f(a) = (f(a))_{(n)}$ [5 points]

Q21) Because $a \circ b = b \circ a$ and f is a homomorphism, $f(a) \cdot f(b) = f(a \circ b) = f(b \circ a) = f(b) \cdot f(a)$ [7 points]

Q22) Because f is a homomorphism, $\theta((aK)(bK)) = \theta((ab)K) = f(ab) = f(a) \cdot f(b) = \theta(aK) \cdot \theta(bK)$ [4 points]
If $\theta(aK) = 1$, then $f(a) = 1$. But then a is in the kernel of f, which we called K. By Q15, this means that $aK = K$. So the kernel of θ is K [3 points]
Suppose that $f(a) = h$. Then $\theta(aK) = f(a) = h$, so the image (range) of θ is H [3 points]