11^{th} Annual Johns Hopkins Math Tournament
Sunday, April 11, 2010
Pitfalls of Algebra: Solutions

(1) (**6**) Find the sum

$$1 + 3 + 5 + \cdots + 97 + 99$$

Answer: 2500
Solution: The desired sum is equal to

$$\begin{aligned} 1 + 3 + \cdots + 97 + 99 &= (1 + 2 + 3 + \cdots + 97 + 98 + 99) - (2 + 4 + \cdots + 98) \\ &= \frac{99(100)}{2} - 2(1 + 2 + \cdots + 49) = 99(50) - 2\frac{49(50)}{2} \\ &= 50^2 = 2500. \end{aligned}$$

(2) (**7**) Let a be any integer in the set $\{0, 1, \ldots, 9\}$. For how many a is the number $3a512a46a$ divisible by 7 and only 7?
Answer: 0
Solution: Note that the sum of the digits is $3 + 5 + 1 + 2 + 4 + 6 + 3a = 21 + 3a$, which is always a multiple of 3. Hence this number is always divisble by 3, so no such a will work.

(3) (**8**) Determine the number of consecutive zeroes at the right end of the decimal expansion of

$$\left(\left(\left(2010^{2009}\right)^{2008}\right)^{\cdot^{\cdot^{\cdot}}}\right)^{1}$$

where $2010 = 2 \cdot 3 \cdot 5 \cdot 67$.
Answer: 2009!
Solution: It suffices to count the number of 5's in the prime factorization. From the factorization given, we see that 2010 is divisible by 10, and contains exactly one 5 in its prime factorization. Hence 2010^{2009} has $2009 \cdot 1 = 2009$ 5's. Repeating in this manner, we see that the entire expression has $1 \cdot 2009 \cdot 2008 \cdots 2 \cdot 1 = 2009!$ 5's in its prime factorization. Hence there are 2009! consecutive zeroes at the end.

(4) (**10**) Find the real number x such that

$$x + 2x^2 + 3x^3 + 4x^4 + \cdots = 30$$

Answer: $\frac{5}{6}$ or $.8333\ldots$
Solution: First note that if $|x| \geq 1$, then the expression will blow up, so we must have $|x| < 1$. We have

$$\sum_{k=1}^{\infty} kx^k = x + 2x^2 + 3x^3 + \cdots = 30 \Longrightarrow \frac{30}{x} = 1 + 2x + 3x^2 + 4x^3 + \cdots.$$

Subtracting the first equation from the second one, we get

$$\frac{30}{x} - 30 = 1 + x + x^2 + x^3 + \cdots.$$

But the right side is simply a geometric series in x and we know its sum is given by $\frac{1}{1-x}$. Hence,

$$30\left(\frac{1-x}{x}\right) = \frac{1}{1-x} \Longrightarrow 30(1-x)^2 = x \Longrightarrow 30x^2 - 61x + 30 = 0.$$

We are left with a simple quadratic, and the quadratic formula yields $x = 1.25$ and $x = .833\ldots$. But we reject the first solution (why?), and hence $x = .8333\ldots = \frac{5}{6}$.

(5) (**11**) Let $a, b, c, d > 0$ be real numbers such that $a + b + c + d = 6$. Find the minimum value of

$$\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{d}\right)^2 + \left(d + \frac{1}{a}\right)^2$$

Answer: $\frac{169}{9}$
Solution: First notice that by the arithmetic-harmonic mean inequality,

$$\frac{a + b + c + d}{4} \geq \frac{4}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}}.$$

Cross-multiplying and using the fact that $a + b + c + d = 6$ by assumption, we have

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \geq \frac{8}{3}.$$

Now, by the root mean square-arithmetic mean inequality,

$$\sqrt{\frac{\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{d}\right)^2+\left(d+\frac{1}{a}\right)^2}{4}} \geq \frac{a+b+c+d+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}{4}$$
$$\geq \frac{6+\frac{8}{3}}{4} = \frac{13}{6}.$$

Squaring both sides and multiplying by four shows that $\frac{169}{9}$ is a lower bound. Setting $a=b=c=d=\frac{3}{2}$ shows that this minimum is actually attained.

(6) (**13**) Compute

$$\frac{1}{e\times\pi}+\frac{1}{\pi\times(2\pi-e)}+\frac{1}{(2\pi-e)\times(3\pi-2e)}+\frac{1}{(3\pi-2e)\times(4\pi-3e)}+\cdots$$

Answer: $\frac{1}{e\times(\pi-e)}$
Solution: Note that if $a\neq b(\neq 0)$ are real numbers, then

$$\frac{1}{a\times b}=\frac{1}{b-a}\left(\frac{1}{a}-\frac{1}{b}\right)$$

Next notice that in each term, the difference between the two terms is always $\pi-e$. That is, $b-a=\pi-e$ in each term. Therefore, when we add together all of the terms using this observation, we get a telescoping sum and find that the desired sum is equal to

$$\frac{1}{\pi-e}\left(\frac{1}{e}-\frac{1}{\pi}+\frac{1}{\pi}-\frac{1}{2\pi-e}+\frac{1}{2\pi-e}+\cdots\right)=\frac{1}{e\times(\pi-e)}$$

(7) (**14**) Find the sum of all integers $0\leq x\leq 100$ such that $f(x)=x^2-3x+27$ is divisible by 37.
Answer: 233
Solution: Applying the quadratic formula, we get that

$$x\equiv\frac{3\pm\sqrt{9-4(27)}}{2}=\frac{3\pm\sqrt{-99}}{2}\pmod{37}$$

We continue to add 37 inside the radical until this yields an integer; adding $4(37)=148$ gives

$$x\equiv\frac{3\pm\sqrt{-99+148}}{2}=\frac{3\pm 7}{2}=5,-2\pmod{37}$$

To recover all of integers between 0 and 100 that work, we simply keep adding 37 to each of these, yielding $\{5,42,79,35,72\}$. Adding these together gives the desired result.

(8) (**15**) What is the largest integer less than or equal to $(\sqrt{3}+\sqrt{2})^6$?
Answer: 969
Solution: If we expand $(\sqrt{3}+\sqrt{2})^6$, we find that the odd powered terms contain $\sqrt{6}$. Since we want an integer, we should get rid of these odd-powered terms. If we expand $(\sqrt{3}-\sqrt{2})^6$, we find that these same terms are negative. Then if we add the two,

$$(\sqrt{3}+\sqrt{2})^6+(\sqrt{3}-\sqrt{2})^6=2\left((\sqrt{3})^6+\binom{6}{2}(\sqrt{3})^4(\sqrt{2})^2+\binom{6}{4}(\sqrt{3})^2(\sqrt{2})^4+(\sqrt{2})^6\right)$$
$$=2(27+15(18+12)+8)=970.$$

Since $(\sqrt{3}-\sqrt{2})<1$, we have

$$(\sqrt{3}+\sqrt{2})^6=970-(\sqrt{3}-\sqrt{2})^6>969.$$

Since $969<(\sqrt{3}-\sqrt{2})^6<970$, the answer is 969.

(9) (**16**) Let A be a set of real numbers such that there always exists x, y in A with the following property:

$$0\leq\frac{x-y}{1+xy}<\frac{1}{\sqrt{3}}$$

What is the minimum number of elements of A such that this holds for any set A?
Answer: 7
Solution: Recall that the range of $\tan x$ for $-\frac{\pi}{2}<x<\frac{\pi}{2}$ is all real numbers. Then if a_i is an element of A, there exists x_i such that $\tan x_i=a_i$. If we divide the interval $(-\frac{\pi}{2},\frac{\pi}{2})$ into 6 equal subintervals, then there are two of the x_i in one of the subintervals if and only if there are 7 or more a_i by the Pigeonhole

Principle. Let these two be $x_j < x_k$. Hence $0 \leq x_k - x_j < \frac{\pi}{6}$. Now $\tan x$ increases on $(0, \frac{\pi}{6})$, so this inequality becomes $0 \leq \tan(x_k - x_j) < \tan(\frac{\pi}{6})$. Using the formula for the tangent-of-difference,

$$0 \leq \frac{\tan x_k - \tan x_j}{1 + \tan x_k \tan x_j} < \frac{1}{\sqrt{3}}$$

Since $\tan x_i = a_i$, this becomes

$$0 \leq \frac{a_k - a_j}{1 + a_k a_j} < \frac{1}{\sqrt{3}}$$

Therefore, we need A to have at least 7 elements for this to always hold.