Johns Hopkins Mathematics Tournament

April 8, 2006

Combinatorics Question Paper

1. Let $P(n)$, for $n \geq 1$, denote the probability that if n coins are flipped, all come up the same. Find S if
$$S = \sum_{k=1}^{\infty} P(k).$$

2. A random point is selected inside a square. What is the probability that of the four perpendiculars dropped from the point to the sides of the square at least three can form a triangle?

3. Four people sit around a round table, each person knowing his two neighbors but not the person across from him. The four get up, walk around, and sit back down at the table in random seats. What is the probability that someone is sitting next to someone he doesn't know?

4. A standard deck of 52 cards consists of four different suits, each with 13 cards. How many 5-card hands are there such that all four suits are present? *Present your answer in prime factorization form.*

5. On average, how many times must a coin be flipped before there are two more tails than heads or two more heads than tails?

6. Find the smallest n so that no matter how n points are placed in a unit square, there exists a pair of points separated by a distance no greater than $\frac{\sqrt{2}}{3}$.

7. Determine the number of distinct primes that divide S, where
$$S = \binom{2000}{1} + 2\binom{2000}{2} + 3\binom{2000}{3} + \ldots + 2000\binom{2000}{2000}.$$

8. Anne writes down the nine consecutive integers from -4 to 4. She then performs a series of operations. In each operation she identifies two numbers that differ by two, decreases the larger by one, and increases the smaller by one so that the two numbers are now equal. After a while she has nine zeros left and stops. How many operations did she perform?