Question One

Chemistry of Mountain Dew [35]

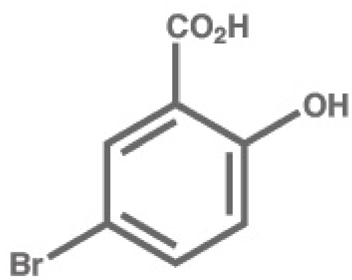
Scientific Mathematics Round

Introduction. Mountain Dew is an extremely popular drink in America. Dominating almost all other sodas with 55 mg of caffeine per can (for comparison, Coca-Cola Classic has only 34 mg), acids and sugars that erode your teeth (dentists have named the resulting tooth decay the "Mountain Dew Mouth"), and even some orange juice, it is a favorite in colleges. It appears on this exam as a tribute to its cultural importance.

(a) [2] Explain[1] why there are $n!$ ways to arrange n different objects in a line.

(b) [3] Explain why there are $(n-1)!$ ways to arrange n different objects in a ring.

(c) [14] Mountain Dew is "syrupy" and gets its density and opacity from *brominated vegetable oil*. A key component of this oil is the molecule *5-bromosalicylic acid* (shown below) which is heavier than normal oils because of the large bromine (Br) atom. Notice that there are three groups attached to the central ring. What is the probability that if we stuck all 3 groups onto the ring at random positions we would obtain 5-bromosalicylic acid?



(d) [3] Surprisingly, the hexagonal ring in the center of the molecule is planar (i.e. fully contained in a plane) in 3D. Those of you who know some chemistry might recall that double bonds (=) are shorter than single bonds (–). A 6-sided ring of such alternating single/double bonds is called a *benzene ring* and it turns out that such rings *do* have the same length on all sides and all the angles are the same. Suppose the side length of this ring is a and that we want to pass a spherical object (such as another molecule) through the center of the ring. What is the maximum radius of this object in terms of a?

(e) [4] The *caffeine content* of a drink is the weight of caffeine (in milligrams) divided by the total volume of the drink (in ounces). Suppose one sample of Mountain Dew has c mg of caffeine and a volume of d oz. Suppose another drink *with equal caffeine content* has e mg of caffeine and a volume of f oz. Call the caffeine content of these drinks K. We know that the caffeine content K' of the mixture of these two samples can be expressed as $xK' = K$.

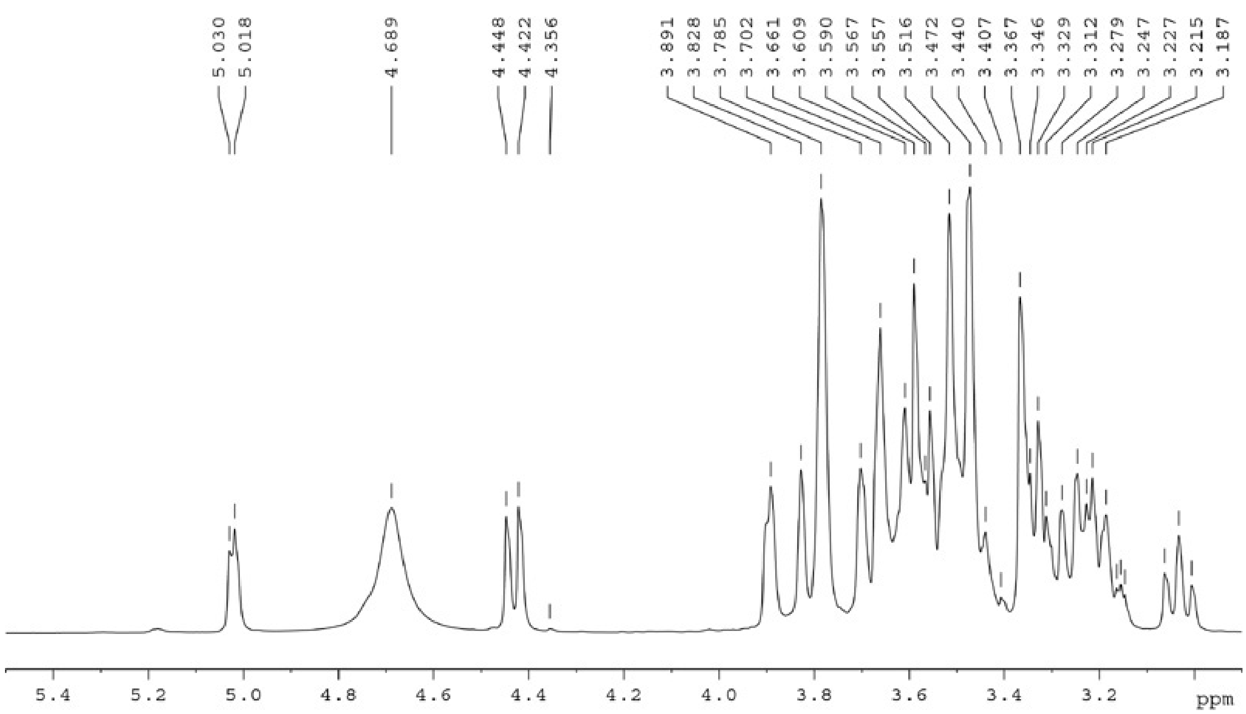
[1] Note: The factorial function $n! = n(n-1)(n-2)\dots 2 \cdot 1$

What is x? In other words, solve for x in the following equation:

$$\frac{c}{d} = \frac{e}{f} = x\frac{c+e}{d+f}.$$

(f) [9] The *H-NMR spectrum* of a sample tells a chemist how many different “types” of hydrogen atoms are in the sample. Although each hydrogen atom is physically the same, in different molecules the hydrogen atoms are surrounded by different atoms and this change in environment causes the hydrogen atoms to be slightly different. An experienced scientist could guess a lot of information about the structure of molecules in a sample just by looking at its H-NMR spectrum.

Shown below is the H-NMR spectrum of Mountain Dew. Each peak corresponds to one type of hydrogen atom, the area under each peak tells us roughly how many hydrogen atoms in the soda are of that type (bigger area means more hydrogen atoms), and the numbers along the *ppm* axis tell us what type of hydrogen atom it is (the *ppm* for each peak is labeled at the top). Hydrogen atoms with $ppm > 4.00$ are likely to be near double bonds of some sort. [4] Forgetting the fact that the peaks have different areas, answer this question: If we take one hydrogen atom from the drink at random, what is the probability that it is near a double bond? [5] Now go back and reconsider that the areas under the peaks as shown in the figure actually do vary. Is the *exact* probability less or greater than the number you calculated in the first part of this question? Why?



Question Two

Designing Plastic Bags [35]

Scientific Mathematics Round

Introduction. Plastic bags are everywhere in our society. They are created by linking *pre-polymers* (relatively small molecules) into long chains called *polymers*, which then group together to make the bag. Pretend you are a chemical engineer designing new plastic bags.

(a) [3] One method to connect the pre-polymers together is to use a *catalyst* (a substance that speeds up the linking process). Our catalyst is a flat square of side 10. Place this square on a Cartesian coordinate system so that its center coincides with the origin and its sides are parallel to the coordinate axes. An analyst determines that pre-polymers will bind only to points on the square whose x and y coordinates are both integers. How many pre-polymers will bind to one side of the catalyst?

(b) [7] Assume you are using the square catalyst of part (a) and that your answer was α pre-polymers. You have constructed a device that can place pre-polymers on the catalyst. It deposits 2 pre-polymers onto the catalyst and then it needs to be reloaded with pre-polymers. However, in the time it takes to get reloaded with another batch of pre-polymers, 1 of the pre-polymers falls off the catalyst. The device takes 1 minute to deposit each pre-polymer and 2 minutes to reload. How long will it take to fully deposit all of the α pre-polymers onto the catalyst surface?

(c) [15] You use the above method to produce a plastic bag and now you intend to test its grocery loading capacity. Assume you have an infinite number of food items, the first weighing 1 kg, the second weighing 2 kg, the third weighing 3 kg, and so on (increasing by 1 kg for each item). Also assume that your bag can hold the first item with probability 1, the second with probability $\frac{1}{2}$, the third with probability $\frac{1}{4}$, and so on (decreasing by a factor of two for each item). What is the expected mass of groceries that your bag can hold?

(d) [10] You know from thermodynamic analysis that the *stretch* limit for your bag is 5; if its stretch ever exceeds 5, it will break. The formula relating the bag's temperature (T_{bag}), stretch (S), mass it is carrying (M), and number of polymers in the bag (N) is $M = \frac{T_{bag}}{N} S$. In one experiment you did on your bag, the bag was cooled to 14^oC and was about to break when it was loaded with 2 kg (you saved it before it actually did break).

Armed with this same bag, you go to the grocery store. Say you have bought 3 kg of produce which was at 4^oC, and you put it in your plastic bag which is at 28^oC. Over time, the temperature of the bag will decrease until it reaches the temperature of the groceries (assume the temperature of the groceries stays at 4^oC). Fourier's Law of Conduction tells us that the temperature changes according to the formula $T_{bag} = 24e^{-t} + 4$, where t is in hours and T_{bag} is in ^{o}C. We start at time $t = 0$, when the bag is just packed with the groceries. How much time do you have to go to your car with the bag before it breaks? *(You need not evaluate any exponential or logarithmic functions).*

Question Three

Friendly Fermies [30]

Scientific Mathematics Round

Introduction. The famous scientist Enrico Fermi is well known for "Fermi questions," puzzling problems he often proposed to his colleagues. Because they are so interesting, people try to mimic his style in similar problems. For every Fermi question, you have to approximate an answer *to the nearest magnitude of 10.* Partial credit is awarded based on how many magnitudes of 10 away from the true answer you were.

Here is an example problem: how many sugar crystals can you stack on each other so that they reach the height of an average person? Well a sugar crystal is a little less than 1 mm long (or 10^{-3} meters) and an average human is somewhere around 1.6 meters. Dividing these quantities, we see that it takes about 1600 sugar crystals, to which the nearest power of 10 is $1000 = 10^3$.

(a) [6] Two people are standing at a starting line, ready to run a 100-meter sprint. An official, standing 50 meters from each of them, points a gun upward and fires it to start the race. One of the runners is blind (but can still run straight) and starts when he *hears the gunshot*, while the other can see and starts when he *sees the flash of the gun* being fired. If both run at 8 meters per second, how many meters does one finish ahead of the other?

(b) [6] How many cubic miles is the volume of Lake Michigan?

(c) [6] How many barbers and hair dressers are there in the United States?

(d) [6] If you were to stack audio CDs from the Earth to the Sun, how many CDs would you need?

(e) [6] What fraction of the United States' area is covered by cars that were sold in 2004?