Johns Hopkins Mathematics Tournament

April 8, 2006

Team Solutions

1. Evaluate S.
$$S = \frac{10000^2 - 1}{\sqrt{10000^2 - 19999}}.$$

   Solution: $\frac{10000^2-1}{\sqrt{10000^2-19999}} = \frac{(10000-1)(10000+1)}{\sqrt{10000^2-2\cdot 10000+1}} = \frac{9999\cdot 10001}{\sqrt{9999^2}} = \boxed{10001}$.

2. Starting on a triangular face of a right triangular prism and allowing moves to only adjacent faces, how many ways can you pass through each of the other four faces and return to the first face in five moves?

   Solution: Each of the other four faces can be traversed in any order, since they are each adjacent to one another. However, the paths that have the other triangular face either first or last have to be thrown out, since the triangular faces are not adjacent. Thus there are $\frac{4!}{2} = \boxed{12}$ paths.

3. Given that
$$\begin{aligned} (a+b)+(b+c)+(c+a) &= 18, \\ \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} &= \frac{5}{9}, \end{aligned}$$
   determine
$$\frac{c}{a+b}+\frac{a}{b+c}+\frac{b}{c+a}.$$

   Solution: We have $a+b+c = 9$. Multiplying that with the second equation gives $\frac{a+b+c}{a+b}+\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a} = 5$, so $\frac{c}{a+b}+\frac{a}{b+c}+\frac{b}{c+a} = (\frac{a+b+c}{a+b}-1)+(\frac{a+b+c}{b+c}-1)+(\frac{a+b+c}{c+a}-1) = \boxed{2}$.

4. Find all primes p such that $2^{p+1}+p^3-p^2-p$ is prime.

   Solution: Since $p = 2$ gives an even number, we'll concentrate on odd p. Looking at the expression mod 3, we find that $2^{p+1} \equiv 1 \bmod 3$ for odd p and $p^2 \equiv 1 \bmod 3$ for p not divisible by 3. Since $p^3-p = (p+1)\cdot p\cdot(p-1)$, a product of three consecutive integers, it is always divisible by three. Thus $2^{p+1}-p^2+(p^3-p) \equiv 1-1+0 \equiv 0 \bmod 3$ for p not divisible by 3. Therefore the only candidate for p is $\boxed{3}$, and $2^{3+1}+3^3-3^2-3 = 31$ is indeed prime.

5. In right triangle ABC with the right angle at A, AF is the median, AH is the altitude, and AE is the angle bisector. If $\angle EAF = 30^o$, find $\angle BAH$ in degrees.

   Solution: Taking $AC > AB$, we have F as the circumcenter of the triangle, so $AF = FC$ and $\angle BAH = 90^o - \angle ABC = \angle ACB = \angle CAF = \angle CAE - \angle EAF = 15^o$. Taking $AC < AB$ yields the complement angle of 75^o, so the accepted answers were $\boxed{15^o \text{ or } 75^o}$.

6. For which integers a does the equation
$$(1-a)(a-x)(x-1) = ax$$
not have two distinct real roots of x?

Solution: First, note that if $a = 1$, the equation is no longer quadratic, so that is one possible answer. Now, for the equation not to have two distinct real roots means the discriminant is non-positive. Expanded, the equation becomes
$$(a-1)x^2 - (a^2+a-1)x + a^2 - a = 0.$$
The discriminant of this is
$$a^4 - 2a^3 + 7a^2 - 6a + 1 = (a^2 - a + 3)^2 - 8.$$
Since $a^2 - a$ is only negative for a between 0 and 1 exclusive, the discriminant is always positive for integral a. Thus the only value of a is $\boxed{1}$.

7. Given that
$$a^2 + b^2 - ab - b + \frac{1}{3} = 0,$$
solve for all (a, b).

Solution: The equation factors as
$$\left(a - \frac{b}{2}\right)^2 + \frac{3}{4}\left(b - \frac{2}{3}\right)^2 = 0,$$
so both squares must equal zero, so that $b = \frac{2}{3}$ and $a = \frac{b}{2} = \frac{1}{3}$. Thus the only answer is $\boxed{\left(\frac{1}{3}, \frac{2}{3}\right)}$.

8. Point E is on side AB of the unit square $ABCD$. F is chosen on $\overline{BC}$ so that $AE = BF$, and G is the intersection of $\overline{DE}$ and $\overline{AF}$. As E varies along side AB, what is the minimum length of BG?

Solution: We have $AD = AB$ and $AE = BF$, so right triangles DAE and ABF are congruent, so that $\angle ADE = \angle BAF = 90^o - \angle DAG$, giving $\angle AGD = 90^o$. Thus point G traces out part of the circle with diameter AD. Letting the midpoint of AD be O, note that by the triangle inequality $OG + BG \geq OB$, so $BG \geq OB - OG = \sqrt{AO^2 + AB^2} - OG = \sqrt{\frac{1}{2}^2 + 1^2} - \frac{1}{2} = \boxed{\frac{\sqrt{5}-1}{2}}$.

9. Sam and Susan are taking turns shooting a basketball. Sam goes first and has probability P of missing any shot, while Susan has probability P of making any shot. What must P be such that Susan has a 50% chance of making the first shot?

Solution: The probability that Susan gets to take, then make her first shot, is $P \cdot P$. The probability that Susan gets to take, then make her second shot, is $P \cdot (1-P) \cdot P \cdot P$. Continuing on, we see that the sum is a geometric series with starting term P^2 and ratio $P(1-P)$, which equals $\frac{P^2}{1-(P(1-P))}$. Equating this with $\frac{1}{2}$ yields $P^2 + P - 1 = 0$, and solving and discarding the extraneous solution gives $\boxed{P = \frac{\sqrt{5}-1}{2}}$.

10. Quadrilateral $ABCD$ has $AB = BC = CD = 7, AD = 13, \angle BCD = 2\angle DAB$, and $\angle ABC = 2\angle CDA$. Find its area.

Solution: Let the intersection of the angle bisectors of $\angle BCD$ and $\angle ABC$ be E. Since $AB = BC = CD, \angle ABE = \angle EBC$, and $\angle BCE = \angle ECD$, $\triangle ABE, \triangle CBE$, and $\triangle CDE$ are congruent. Since $\angle CDA = \frac{1}{2}\angle ABC = \angle CBE = \angle CDE, E$ lies on $\overline{AD}$, so $BE+EC = ED+AE = AD = 13$. Also, the three congruent angles of $\angle AEB, \angle BEC$, and $\angle CED$ sum to 180^o, so $\angle BEC = 60^o$. Applying the Law of Cosines to triangle BEC gives $BE^2 + (13 - BE)^2 - \frac{1}{2}\cos 60^o \cdot BE(13 - BE) = 7^2$, which has a solution of $BE = 8$ or 5. Thus, $[ABCD] = 3[BCE] = 3 \cdot \frac{1}{2}\sin 60^o \cdot 8 \cdot 5 = \boxed{30\sqrt{3}}$.