Johns Hopkins Mathematics Tournament

April 8, 2006

Geometry Solutions

1. Unit square $ZINC$ is constructed in the interior of hexagon $CARBON$. What is the area of triangle BIO?

   Solution: Let M be the midpoint of side BN. Since BON is a $30^o - 30^o - 120^o$ triangle, $BN = \sqrt{3}$ and $OM = \frac{1}{2}$. Therefore, $[BIO] = \frac{1}{2}BI \cdot OM = \boxed{(\sqrt{3}-1)/4}$.

2. If two altitudes of a triangle have length 12 and 4, what integral lengths can the third altitude attain?

   Solution: If we choose the area of the triangle to be an easy-to-work-with number like 6, the sides of the triangle are 1, 3, and $\frac{12}{x}$. Since the sum of any two sides of a triangle is greater than the third side, we must have $1+3 > \frac{12}{x}$ and $1 + \frac{12}{x} > 3$, so that $2 < \frac{12}{x} < 4$, or $3 < x < 6$. The only integers satisfying this are $\boxed{\text{4 and 5}}$.

3. Rectangle $ABCD$ is folded in half so that the vertices D and B coincide, creating the crease $\overline{EF}$, with E on $\overline{AD}$ and F on $\overline{BC}$. Let O be the midpoint of $\overline{EF}$. If triangles DOC and DCF are congruent, what is the ratio $BC : CD$?

   Solution: Since O is the midpoint of the rectangle, $BD = DO + BO = 2DO = 2DC$. Thus BCD is a $30^o - 60^o - 90^o$ triangle and $BC : CD = \boxed{\sqrt{3} : 1}$.

4. The square $DEFG$ is contained in equilateral triangle ABC, with E on $\overline{AC}$, G on $\overline{AD}$, and F as the midpoint of $\overline{BC}$. Find AD if $DE = 6$.

   Solution: Let the midpoint of $\overline{DF}$ be H. $\overline{DF} \perp \overline{EH}$, so AHE is a $30^o - 60^o - 90^o$ triangle. $EH = DH = \frac{1}{2}DF = \frac{1}{2} \cdot 6\sqrt{2} = 3\sqrt{2}$, and $AH = \sqrt{3} \cdot EH = 3\sqrt{6}$, so $AD = AH - DH = \boxed{3\sqrt{6} - 3\sqrt{2}}$.

5. An ant is on the bottom edge of a right circular cone with base area π and slant length 6. What is the shortest distance that the ant has to travel to loop around the cone and come back to its starting position?

   Solution: The radius of the cone is 1, so its circumference is 2π. Unfolding the cone results in a sector with radius 6 and circular circumference 2π. Label the center O and the other two vertices A and B. The full circle has a circumference of 12π, so the sector is a $\frac{2\pi}{12\pi} \cdot 360^o = 60^o$ sector, so that ABO is an equilateral triangle and the shortest distance is $AB = \boxed{6}$.

6. A right cylinder is inscribed in a right circular cone with height 2 and radius 2 so that the cylinder's bottom base sits on the cone's base. What is the maximum possible surface area of the cylinder?

   Solution: That the height and radius of the cone are both equal to 2 means that the height and the radius of the cylinder add up to 2. Letting h and r be the height and radius of the cylinder, respectively, the surface area $= 2\cdot\pi r^2+2\pi rh = 2\pi r(r+h) = 4\pi r$. The maximum r can be is 2, so the maximum surface area is $\boxed{8\pi}$. Note that the cylinder is a degenerate flat disk.

7. AD is the angle bisector of the right triangle ABC with $\angle ABC = 60^o$ and $\angle BCA = 90^o$. E is chosen on $\overline{AB}$ so that the line parallel to $\overline{DE}$ through C bisects $\overline{AE}$. Find $\angle EDB$ in degrees.

   Solution: Let the midpoint of $\overline{AE}$ be F and the intersection of $\overline{ED}$ and $\overline{AC}$ be A'. Since $\triangle ACF \sim \triangle AA'E$, $\frac{AF}{FE} = \frac{AC}{CA'}$. Therefore $\triangle ACD \cong \triangle A'CD$, so $\angle EDB = \angle A'DC = \angle ADC = 90^o - \angle DAC = 90^o - \frac{1}{2}\cdot 30^o = \boxed{75^o}$.

8. Circles P, Q, and R are externally tangent to one another. The external tangent of P and Q that does not intersect R intersects P and Q at P_Q and Q_P, respectively. Q_R, R_Q, R_P, and P_R are defined similarly. If the radius of Q is 4 and $\overline{Q_PP_Q} \parallel \overline{R_QQ_R}$, compute R_PP_R.

   Solution 1: Let the radii of circles P and R be p and r, respectively, and let the three centers be O_P, O_Q, and O_R. Since $\overline{Q_PP_Q} \parallel \overline{R_QQ_R}$, Q lies on $\overline{Q_PQ_R}$. Let the perpendicular from P to $\overline{Q_PQ}$ be PA, and let the perpendicular from R to $\overline{PA}$ be $\overline{RB}$. Since Q_PP_QAP is a rectangle, We have $Q_PP_Q = PA = \sqrt{PQ^2 - AQ^2} = \sqrt{(4+p)^2 - (4-p)^2} = 4\sqrt{p}$. Similarly, $Q_RR_Q = 4\sqrt{r}$ and $R_PP_R = 2\sqrt{pr}$. Then, in right triangle PBR, $0 = PR^2 - PB^2 - BR^2 = (p+r)^2 - (Q_RR_Q - Q_PP_Q)^2 - (8-p-r)^2 = (p+r)^2 - (4\sqrt{r} - 4\sqrt{p})^2 - (8-p-r)^2 = 32\sqrt{pr} - 64$, so $\sqrt{pr} = 2$ and $R_PP_R = \boxed{4}$.

   Solution 2: Since we can choose any circles P and R, we can let them have the same radii. Since they are sandwiched between two parallel lines that are 8 units apart, their radii equal 2. We find that R_PP_RPR is a rectangle, and $R_PP_R = PR = \boxed{4}$.