Geometry Solutions

1. Unit square $ZINC$ is constructed in the interior of hexagon $CARBON$. What is the area of triangle $BIO$?

Solution: Let $M$ be the midpoint of side $BN$. Since $BON$ is a $30^\circ - 30^\circ - 120^\circ$ triangle, $BN = \sqrt{3}$ and $OM = \frac{1}{2}$. Therefore, $[BIO] = \frac{1}{2}BI \cdot OM = \frac{(\sqrt{3} - 1)}{4}$.

2. If two altitudes of a triangle have length 12 and 4, what integral lengths can the third altitude attain?

Solution: If we choose the area of the triangle to be an easy-to-work-with number like 6, the sides of the triangle are 1, 3, and $\frac{12}{x}$. Since the sum of any two sides of a triangle is greater than the third side, we must have $1 + 3 > \frac{12}{x}$ and $1 + \frac{12}{x} > 3$, so that $2 < \frac{12}{x} < 4$, or $3 < x < 6$. The only integers satisfying this are 4 and 5.

3. Rectangle $ABCD$ is folded in half so that the vertices $D$ and $B$ coincide, creating the crease $EF$, with $E$ on $AD$ and $F$ on $BC$. Let $O$ be the midpoint of $EF$. If triangles $DOC$ and $DCF$ are congruent, what is the ratio $BC : CD$?

Solution: Since $O$ is the midpoint of the rectangle, $BD = DO + BO = 2DO = 2DC$. Thus $BCD$ is a $30^\circ - 60^\circ - 90^\circ$ triangle and $BC : CD = \sqrt{3} : 1$.

4. The square $DEFG$ is contained in equilateral triangle $ABC$, with $E$ on $AC$, $G$ on $AD$, and $F$ as the midpoint of $BC$. Find $AD$ if $DE = 6$.

Solution: Let the midpoint of $DF$ be $H$. $DF \perp EH$, so $AHE$ is a $30^\circ - 60^\circ - 90^\circ$ triangle. $EH = DH = \frac{1}{2}DF = \frac{1}{2} \cdot 6\sqrt{2} = 3\sqrt{2}$, and $AH = \sqrt{3 \cdot EH} = 3\sqrt{6}$, so $AD = AH - DH = 3\sqrt{6} - 3\sqrt{2}$.

5. An ant is on the bottom edge of a right circular cone with base area $\pi$ and slant length 6. What is the shortest distance that the ant has to travel to loop around the cone and come back to its starting position?

Solution: The radius of the cone is 1, so its circumference is $2\pi$. Unfolding the cone results in a sector with radius 6 and circular circumference $2\pi$. Label the center $O$ and the other two vertices $A$ and $B$. The full circle has a circumference of $12\pi$, so the sector is a $\frac{2\pi}{12\pi} \cdot 360^\circ = 60^\circ$ sector, so that $ABO$ is an equilateral triangle and the shortest distance is $AB = 6$.

6. A right cylinder is inscribed in a right circular cone with height 2 and radius 2 so that the cylinder’s bottom base sits on the cone’s base. What is the maximum possible surface area of the cylinder?

Solution: That the height and radius of the cone are both equal to 2 means that the height and the radius of the cylinder add up to 2. Letting $h$ and $r$ be the height and
radius of the cylinder, respectively, the surface area = \(2\pi r^2 + 2\pi rh = 2\pi r(r + h) = 4\pi r\). The maximum \(r\) can be is 2, so the maximum surface area is \(8\pi\). Note that the cylinder is a degenerate flat disk.

7. \(AD\) is the angle bisector of the right triangle \(ABC\) with \(\angle ABC = 60^\circ\) and \(\angle BCA = 90^\circ\). \(E\) is chosen on \(AB\) so that the line parallel to \(DE\) through \(C\) bisects \(AE\). Find \(\angle EDB\) in degrees.

**Solution:** Let the midpoint of \(AE\) be \(F\) and the intersection of \(ED\) and \(AC\) be \(A'\). Since \(\triangle ACF \sim \triangle AA'E\), \(\frac{AF}{FE} = \frac{AC}{CA'}\). Therefore \(\triangle ACD \sim \triangle A'CD\), so \(\angle EDB = \angle A'DC = \angle ADC = 90^\circ - \angle DAC = 90^\circ - \frac{1}{2} \cdot 30^\circ = 75^\circ\).

8. Circles \(P\), \(Q\), and \(R\) are externally tangent to one another. The external tangent of \(P\) and \(Q\) that does not intersect \(R\) intersects \(P\) and \(Q\) at \(PQ\) and \(Q\). \(RP\), \(RQ\), \(RP\), and \(RP\) are defined similarly. If the radius of \(Q\) is 4 and \(QP\parallel QR\), compute \(RP\).

**Solution 1:** Let the radii of circles \(P\) and \(R\) be \(p\) and \(r\), respectively, and let the three centers be \(O_P\), \(O_Q\), and \(O_R\). Since \(QP\parallel QR\), \(Q\) lies on \(QR\). Let the perpendicular from \(P\) to \(QP\) be \(PA\), and let the perpendicular from \(R\) to \(PA\) be \(RB\). Since \(QP\parallel AP\) is a rectangle, We have \(QP = PA = \sqrt{PQ^2 - AQ^2} = \sqrt{(4 + p)^2 - (4 - p)^2} = 4\sqrt{p}\). Similarly, \(RP = 4\sqrt{r}\). Then, in right triangle \(PBR\), \(0 = PR^2 - PB^2 - BR^2 = (p + r)^2 - (QR - RPQ)^2 - (8 - p - r)^2 = (p + r)^2 - (4\sqrt{r} - 4\sqrt{p})^2 - (8 - p - r)^2 = 32\sqrt{p} - 64\), so \(\sqrt{p} = 2\) and \(RP = 4\).

**Solution 2:** Since we can choose any circles \(P\) and \(R\), we can let them have the same radii. Since they are sandwiched between two parallel lines that are 8 units apart, their radii equal 2. We find that \(RP = PR = 4\).