Johns Hopkins Mathematics Tournament

April 8, 2006

Combinatorics Question Paper

1. Let $P(n)$, for $n \geq 1$, denote the probability that if n coins are flipped, all come up the same. Find S if
$$S = \sum_{k=1}^{\infty} P(k).$$

   Solution: $P(n)$ is simply twice the probability that all n coins come up heads, which is $\frac{1}{2^n}$. Thus, $P(n) = \frac{1}{2^{n-1}}$, and $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \boxed{2}$.

2. A random point is selected inside a square. What is the probability that of the four perpendiculars dropped from the point to the sides of the square at least three can form a triangle?

   Solution: Let the perpendiculars be labelled a, b, c, and d, with $a \leq b \leq c \leq d$. Since the two horizontal altitudes and the two vertical altitudes each sum to the side length of the square, $a + d = b + c > b$, so the altitudes a, b, and d, among others, can form a triangle. Since this is true of any square, the probability is $\boxed{1}$.

3. Four people sit around a round table, each person knowing his two neighbors but not the person across from him. The four get up, walk around, and sit back down at the table in random seats. What is the probability that someone is sitting next to someone he doesn't know?

   Solution: Without loss of generality, let one of the people sit at the same seat before and after walking around. Then of the 3! possible ways for the other three people to sit down, only the original seating and its mirror image has everyone sitting next to their original neighbors. Thus the probability of that not happening is $1 - \frac{2}{6} = \boxed{\frac{2}{3}}$.

4. A standard deck of 52 cards consists of four different suits, each with 13 cards. How many 5-card hands are there such that all four suits are present? *Present your answer in prime factorization form.*

   Solution: There are 13^4 ways to do this with a 4-card hand, since for each suit we select one card. Any of the remaining 48 cards can be added for a fifth card. However, this counts twice the number of ways to select two cards from the suit of the fifth card, so the answer is $13^4 \cdot 48 \cdot \frac{1}{2} = \boxed{2^4 \cdot 3 \cdot 13^4}$.

5. On average, how many times must a coin be flipped before there are two more tails than heads or two more heads than tails?

   Solution: Let the answer be S. After two flips, either the two flips were the same of they they were different, both occurring with probability $\frac{1}{2}$. In the first case, the condition has been fulfilled and it took two flips. In the second case, we are back to where we started, needing S more flips to fulfill the condition. Thus $S = \frac{1}{2} \cdot 2 + \frac{1}{2}(S+2)$, giving $\boxed{S = 4 \text{ times}}$.

6. Find the smallest n so that no matter how n points are placed in a unit square, there exists a pair of points separated by a distance no greater than $\frac{\sqrt{2}}{3}$.

   Solution: By dividing the square into nine $\frac{1}{3}$ by $\frac{1}{3}$ squares, we see that the maximum value of n is 10, since then at least one square will have at least two points in it, and the maximum distance in such a square is the diagonal, with length $\frac{\sqrt{2}}{3}$. By placing four points extremely near the corners of the square, four points extremely near the midpoint of each side, and one point in the center of the square, we see that nine points can be placed with minimum distance between any two slightly less than $\frac{1}{2}$, which is greater than $\frac{\sqrt{2}}{3}$, and so $\boxed{10}$ is also the minimum.

7. Determine the number of distinct primes that divide S, where

$$S = \binom{2000}{1} + 2\binom{2000}{2} + 3\binom{2000}{3} + \ldots + 2000\binom{2000}{2000}.$$

   Solution: Since $n\binom{2000}{n} = \frac{n \cdot 2000!}{n! \cdot (2000-n)!} = \frac{2000 \cdot 1999!}{(n-1)! \cdot (1999-(n-1))!} = 2000\binom{1999}{n-1}$,

$$\begin{aligned} S &= 2000\left(\binom{1999}{0} + \binom{1999}{1} + \binom{1999}{2} + \ldots + \binom{1999}{1999}\right) \\ &= 2000 \cdot 2^{1999} \\ &= 2^{2003} \cdot 5^3, \end{aligned}$$

   which has $\boxed{2}$ distinct prime factors.

8. Anne writes down the nine consecutive integers from -4 to 4. She then performs a series of operations. In each operation she identifies two numbers that differ by two, decreases the larger by one, and increases the smaller by one so that the two numbers are now equal. After a while she has nine zeros left and stops. How many operations did she perform?

   Solution: Call the two numbers that Anne changes during an operation $a+1$ and $a-1$. Before the operation, the sum of the squares of these two is $a^2+2a+1+a^2-2a+1 = 2a^2+2$, whereas after the operation the sum of the squares is simply $2a^2$. Since the other numbers do not change, we see that the sum of the squares of all nine numbers goes down by two during every operation. Because $((-4)^2+(-3)^2+(-2)^2+(-1)^2+0^2+1^2+2^2+3^2+4^2)-(0^2+0^2+0^2+0^2+0^2+0^2+0^2+0^2+0^2) = 60$, Anne performed $\frac{60}{2} = \boxed{30 \text{ operations}}$.