Question One

# Proteins and Graph Theory [50]

*Scientific Mathematics Round*

(b) Loosely worded, an attempt to draw a protein-existence schedule that is consistent with this graph leads to some contradiction in that two of 3, 5, and 7 would have to overlap or two of 2, 4, and 6 would have to.

(c) Draw a graph as follows: the vertices are the proteins and an edge connects two proteins if those proteins can give markers to one another. Now draw $A$ next to protein 1, $B$ next to protein 3, and $C$ next to protein 5. Imagine moving the markers around the graph, hopping from vertex to adjacent vertex. Can some sequence of moves end up with $A$ on 2, $B$ on 4, and $C$ on 6? A little tinkering shows not, because markers cannot cross one another; in other words, the "clockwise orientation" of the three pieces cannot change and the end configuration is cyclically incompatible with the initial.

Question Two

# Radioactive Decay [50]

*Scientific Mathematics Round*

(a) As $A$ is the number of neutrons and protons and $Z$ is the number of protons, $\boxed{A - Z}$ is the number of neutrons.

(b) Given initial state $(A, Z)$, alpha decay yields $\boxed{(A - 4, Z - 2)}$, proton emission culminates in $\boxed{(A - 1, Z - 1)}$, and neutron emission leads to $\boxed{(A - 1, Z)}$.

(c) We can get the shortest chain if we max out the number of alpha decays. Observe that we can do at most 5 alpha decays, taking our nucleus to $(218, 82)$. From there, 12 neutron emissions take us to the end state, so that the shortest chain has $12 + 5 = \boxed{17}$ decay steps.

(d) Note that we need exactly 10 proton emissions to get $Z = 82$, and then we need 22 neutron emissions to get to $(206, 82)$. The order of these decay steps however can vary. Of the 32 total steps, we can choose any 10 to be proton emissions, and then the other 22 are forced to be neutron emissions. The total number of decay paths is thus the number of ways to choose 10 of the 32 steps to be proton emissions, which is $\boxed{\binom{32}{10}}$.

(e) Plugging in $\lambda = 0.1$ and $T = 10$ shows that $P(k) = \frac{e^{-1}}{k!}$. The probability that we see more than 2 decays is the complement of the probability that we say no more than 2 decays which is $P(0) + P(1) + P(2)$. The desired answer is thus

$$\begin{aligned} 1 - P(0) - P(1) - P(2) &= 1 - e^{-1}\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!}\right) \\ &= \boxed{1 - 2.5e^{-1}}. \end{aligned}$$

(f) Use the definition of $T_{1/2}$ and the provided bulk decay formula.

$$\begin{aligned} N(t) &= N_0 e^{-\lambda t} \\ \frac{N_0}{2} &= N_0 e^{-\lambda T_{1/2}} \end{aligned}$$

$$
\begin{aligned}
\frac{1}{2} &= e^{-\lambda T_{1/2}} \\
\ln\left(\frac{1}{2}\right) &= -\lambda T_{1/2} \\
\frac{\ln 2}{\lambda} &= T_{1/2},
\end{aligned}
$$

as desired.