Johns Hopkins Mathematics Tournament

April 8, 2006

Algebra Solutions

1. Determine the units digit of $87^{65} + 43^{21}$.

   Solution: Since all fourth powers of odd numbers that aren't multiples of five end in 1, $87^{65} + 43^{21} = 87^{64} \cdot 87 + 43^{20} \cdot 43 \equiv 87 + 43 = 130 \equiv \boxed{0} \bmod 10$.

2. How many integer pairs (a, b) for $-5 \leq a, b \leq 5$ are there such that $ax^2 + 10x + b = 0$ has two distinct real roots?

   Solution: For the equation to have two distinct real roots, the discriminant $= 100-4ab$ must be positive, which means $ab < 25$. This is true for all (a, b) for $-5 \leq a, b \leq 5$ except for $(5, 5)$ and $(-5, -5)$, for which equality occurs. Also, a cannot be 0, since then the equation is no longer quadratic. Since there are 11 possible values for b and 10 for a, the total number of pairs equals $11 \cdot 10 - 2 = \boxed{108}$.

3. Find all x in $[0, \pi]$ inclusive so that
$$\sin^2 x + \csc^2 x + \cos^2 x = \cot^2 x + \sec^2 x.$$

   Solution: The two identities $\sin^2 x + \cos^2 x = 1$ and $\cot^2 x + 1 = \csc^2 x = 1$ reduce the equation to $\sec^2 x = 2$, or $\cos x = \pm\frac{\sqrt{2}}{2}$. The only values of x in $[0, \pi]$ inclusive that work are $\boxed{x = \frac{\pi}{4} \text{ and } \frac{3\pi}{4}}$.

4. Solve for all values of x.
$$x^{\log x} = 1000x^2.$$

   Solution: Taking the log of both sides, we get $\log x \cdot \log x = \log 1000x^2 = 3 + 2\log x$. This is a quadratic in $\log x$, and we can factor it to get $(\log x - 3)(\log x + 1) = 0$, so $\log x = -1$ or 3, and $\boxed{x = 1000 \text{ or } 0.1}$.

5. Given that the number $m3mmmmmmm$ is a prime for at least one digit m, find all such m.

   Solution: Any even value of m will make an even number, and 535555555 is divisible by 5. Both 3 and 9 make the number divisible by 3, and the divisibility test for 11 shows that 11 divides it when m equals 7. The only value of m left is 1, and since the number is prime for at least one value of m, m must equal $\boxed{1}$.

6. A circular disk of gradually decreasing radius slides down a pit defined by the equation $y = x^2$, maintaining two points of contact with the pit. What is its radius when the two points of contact coalesce into one?

   Solution: The coalescing point must be at $(0, 0)$ by symmetry. Thus, the circle is of the form $(y-r)^2 + x^2 = r^2$. Subtracting the bottom half of the circle from the parabola gives $r - \sqrt{r^2 - x^2} - x^2$. Equating this with 0 and solving gives $x^2(x^2 - 2r + 1) = 0$. Thus, when $2r = 1$ or $\boxed{r = \frac{1}{2}}$, the equation becomes $x^4 = 0$ and the circle only contacts the parabola at $(0, 0)$.

7. Between which two consecutive integers does S lie?

$$S = \sqrt{18 + \sqrt{18^2 + \sqrt{18^3 + ...}}}$$

Solution: First, letting $T = \sqrt{1 + \sqrt{1 + \sqrt{1 + ...}}}$, we have $T = \sqrt{1+T}$, giving $T = \frac{\sqrt{5}+1}{2}$.

We have $S > \sqrt{18 + \sqrt{18^2}} = 6$. Also, $S < \sqrt{18^1 + \sqrt{18^2 + \sqrt{18^4 + \sqrt{18^8 + ...}}}} = \sqrt{18} \cdot T = \frac{\sqrt{90}+\sqrt{18}}{2} < \frac{\sqrt{90.25}+\sqrt{20.25}}{2} = \frac{9.5+4.5}{2} = 7$. Thus, S lies between $\boxed{\text{6 and 7}}$.

8. If the range of
$$f(x, y, z) = \frac{x}{x+y} + \frac{y}{y+z} + \frac{z}{z+x}$$
for positive x, y, and z is (A, B) exclusive, find $A + B$.

Solution: Note that $f(x, y, z) + f(z, y, x) = (\frac{x}{x+y} + \frac{y}{y+z} + \frac{z}{z+x}) + (\frac{y}{x+y} + \frac{z}{y+z} + \frac{x}{z+x}) = 3$. Thus, the range of the function is symmetric with respect to 1.5 and $A + B = \boxed{3}$.