Johns Hopkins Mathematics Tournament

April 23, 2005

Team Solutions

1. The sum is equal to

$$\begin{aligned}
&= f(1)+f(2)+f(3)+f(4)+\dots \\
&= [(\frac{1}{2})^1-(\frac{1}{2})^2]+[(\frac{1}{2})^2-(\frac{1}{2})^3]+[(\frac{1}{2})^3-(\frac{1}{2})^4]+\dots \\
&= (\frac{1}{2})^1-[(\frac{1}{2})^2-(\frac{1}{2})^2]-[(\frac{1}{2})^3-(\frac{1}{2})^3]-\dots \\
&= \boxed{\tfrac{1}{2}}.
\end{aligned}$$

2. The shape is that of the triangle bounded by $(0,-4),(4,0)$, and $(-4,0)$ with the triangle bounded by $(0,-2),(2,0)$, and $(-2,0)$ taken out of it. Thus the area is $(8)(4)(\frac{1}{2})-(4)(2)(\frac{1}{2})=\boxed{12}$.

3. Let the triangle be ABC and the center of the circle be O. Draw a radius perpendicular to AB, letting it hit the circle at P. Let D be the intersection of OP and AB. DP is the diameter of the circle in question. Since $\angle POB=\frac{1}{2}\angle AOB=60^o$ and $OP=OB$, POB is an equilateral triangle and $\triangle DBP$ is a $30^o-60^o-90^o$ triangle, so $\frac{DB}{DP}=\sqrt{3}$. Finally, $DB=\frac{1}{2}AB=3$, so $DP=\frac{DB}{\sqrt{3}}=\boxed{\sqrt{3}}$.

4. Multiplying by $\cos x$, rearranging, and factoring out $\sin x$ gives $\sin x(\cos x-\sin x-1)=0$, so either $\sin x=0$ or $\cos x-\sin x=1$. The first equation gives $x=\pi$, since 0 and 2π are not in the interval. Squaring the second equation gives $\cos^2 x+\sin^2 x-2\cos x\sin x=1$. Since $\cos^2 x+\sin^2 x=1$, this equation reduces to $\cos x\sin x=0$. Since we already took care of $\sin x=0$, we solve $\cos x=0$ to get $x=\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$. However, $\tan\frac{\pi}{2}$ and $\tan\frac{3\pi}{2}$ are undefined, so the only answer is $x=\boxed{\pi}$.

5. The grid contains 7 horizontal and 7 vertical lines. Note that by specifying two horizontal and two vertical lines we can define a unique rectangle. Since there are $\binom{7}{2}=21$ ways to choose two horizontal lines and $\binom{7}{2}=21$ ways to choose two vertical lines, the answer is $21^2=\boxed{441}$.

6. The lateral surface area is that of a sector of a circle with the radius of the slant length of the cone which is $\sqrt{3^2+4^2}=5$. The circumference of the full circle is $(2\pi)(5)=10\pi$. The arc length of the sector is the circumference of the cone's base circle, $(2\pi)(3)=6\pi$. Thus, the sector is $\frac{6}{10}=\frac{3}{5}$ of the circle, and its area is $(\frac{3}{5})(\pi)(5^2)=\boxed{15\pi}$.

7. There can't have been four or more 10's because the mode is 7, meaning there had to be more 7's than 10's (five 7's and four 10's doesn't average to 8). If there were three

10's there needs to be at least four 7's. There can't be five 7's because the median, the fifth largest number out of the 9, is an 8. If there are four 7's, the numbers would be $(7, 7, 7, 7, 8, x, 10, 10, 10)$, with x either 8 or 9. However, neither of these "nonuplets" average to 8. Thus, there were at most $\boxed{\text{two}}$ 10's, and a possible score distribution is $(7, 7, 7, 7, 8, 8, 8, 10, 10)$.

8. Let the expression equal x. Then we have $2i + \frac{1}{x} = x$. Multiplying by x gives $x^2 - 2ix - 1 = 0$. The quadratic formula gives

$$\begin{aligned} x &= \frac{2i \pm \sqrt{(-2i)^2 - (4)(1)(-1)}}{(2)(1)} \\ &= \frac{2i \pm \sqrt{0}}{2} \\ &= \boxed{i}. \end{aligned}$$

9. The statement of the problem means that $x^3 - px^2 + px - 6 = (x-p)(x-q)(x-r) = 0$. Thus, letting $x = 1$ yields

$$\begin{aligned} (1-p)(1-q)(1-r) &= 1^3 - p(1^2) + p(1) - 6 \\ &= \boxed{-5}. \end{aligned}$$

10. If the device says Gilnor is a thief, there are two possibilities: Gilnor is a thief and the device is right, or Gilnor isn't a thief and the device is wrong. The first occurs with probability of $(10\%)(90\%)$ and the second with probability of $(1 - 10\%)(1 - 90\%) = (10\%)(90\%)$. Since each is equally likely, the probability is $\boxed{\frac{1}{2}}$.