Johns Hopkins Mathematics Tournament

April 23, 2005

Geometry Solutions

1. Perpendicularly bisecting AC gives a diameter of the large circle. The two pieces of the diameter, separated by AC, happen to be the diameters of the two smaller circles. Thus, the sum of the radii of the smaller circles is $\boxed{\frac{23}{2}}$.

2. $\triangle ABC$ is a $30^o - 60^o - 90^o$ triangle, also AE, which equals PS, the width of the rectangle, is $AB\cdot\sqrt{3} = \sqrt{3}$. $\triangle PAF$ and $\triangle BQC$ are also $30^o - 60^o - 90^o$ triangles, giving $PA = \frac{1}{2}\cdot AF = BQ = \frac{1}{2}$, so $PQ = PA + AB + PQ = 2$, and the area of $PQRS$ is $2\cdot\sqrt{3} = \boxed{2\sqrt{3}}$.

3. Let BD be the perpendicular to AC, with D on the extension of AC. Since $\angle BDA = 90^o$ and $\angle DAB = 180^o - \angle BAC = 45^o$, $\triangle BAD$ is a $45^o - 45^o - 90^o$ triangle. Thus, $BD = \frac{AB}{\sqrt{2}} = \sqrt{2}$ and the area of $\triangle ABC$ is $\frac{1}{2}\cdot AC\cdot BD = \boxed{\sqrt{2}}$.

4. Drop perpendiculars AF and AE to CD, with E and F on CD. $\frac{1}{2}\cdot(AB + CD)\cdot AF = 36$, so $AF = \frac{36}{\frac{1}{2}\cdot 18} = 4$. Since $ABEF$ is a rectangle, $CE = DF = \frac{CD-AB}{2} = 3$, and $BC = \sqrt{BE^2 + CE^2} = \boxed{5}$.

5. Since the rectangle $HJKL$ is rearranged from $\triangle ABC$, they have the same area, which is $\sqrt{3}\cdot\frac{AC^2}{4} = \sqrt{3}\cdot\frac{(AE+EC)^2}{4} = \boxed{16\sqrt{3}}$.

6. Because DF is parallel to BC, $\triangle ADF$ $\triangle ABC$, so $\frac{AD}{DF} = \frac{AB}{BC}$. Solving for BC gives $BC = AB\cdot\frac{DF}{AD} = (AD + BD)\cdot\frac{DF}{AD} = (25 + 10)\cdot\frac{10}{25} = \boxed{14}$.

7. Let the center of the circle be O. The region in question consists of $\triangle AOC$, $\triangle AOB$ and the minor sector BOC. Each of $\angle BOA$, $\angle AOC$, and $\angle COB$ is 120^o, so the total area is $\frac{1}{2}\cdot OC\cdot OA\cdot\sin\angle COA + \frac{1}{2}\cdot OA\cdot OB\cdot\sin\angle AOB + \frac{120^o}{360^o}\cdot$(area of circle) $= \frac{18\sqrt{3}}{2} + \frac{18\sqrt{3}}{2} + \frac{1}{3}\cdot 36 = \boxed{18\sqrt{3} + 12\pi}$.

8. Without loss of generality, let $AB > AC$. $\triangle ABC$ and $\triangle DEC$ are both $30^o - 60^o - 90^o$ triangles, $2 = \frac{1}{4}BC = AC = AD + DC = ED + DC = ED + \frac{ED}{\sqrt{3}}$. Solving for ED gives $ED = \frac{2}{\frac{1}{\sqrt{3}+1}} = \boxed{3 - \sqrt{3}}$.

9. Let the square be $ABCD$, with AB on the hemisphere's diameter, and let O be the midpoint of the diameter. Then we have $OB = \frac{1}{2}$ and $BC = 1$, so the radius is $OC = \sqrt{OB^2 + BC^2} = \frac{\sqrt{5}}{2}$. Thus, the perimeter is $\frac{\pi\sqrt{5}}{2} + \frac{2\sqrt{5}}{2} = \boxed{\sqrt{5} + \frac{\pi\sqrt{5}}{2}}$.

10. Since $\triangle DEC$ is an isosceles triangle, we have $\sqrt{2} \cdot ED = CD = \sqrt{2} \cdot AD$, so $ED = AD$ and $\triangle DAE$ is isosceles. Also, since $\angle EDC = 45^o$, $\angle ADE = 90^o - \angle EDC = 45^o$. Thus, $\angle AED = \frac{180^o - \angle ADE}{2} = \angle BEC$ by symmetry. Finally, $\angle AEB = 360^o - \angle DEA - \angle CED - \angle BEC = 360^o - \frac{2 \cdot (180^o - \angle ADE)}{2} - 90^o = \boxed{135^o}$.