Johns Hopkins Mathematics Tournament

April 23, 2005

Combinatorics Solutions

1. The least common multiple of $1, 2, 3, 4$, and 5 is 60, so we are looking for the number of three-digit multiples of 60. Since $1000/60 = 16\frac{2}{3}$, giving 16 multiples of 60 under 1000, and 60 has 2 digits, the answer is $16 - 1 = \boxed{15}$.

2. The number of ways to choose two fish is $\binom{16}{2} = 120$. To choose two fish of opposite gender we need one male fish (6 choices) and one female fish (10 choices), giving a total of 60 choices, so the answer is $\frac{60}{120} = \boxed{\frac{1}{2}}$.

3. There are 13 odd and 12 even numbers. The max of 13 and 12 is 13, so choosing $\boxed{14}$ numbers will guarantee at least one of each.

4. Most numbers have an even number of factors, since they come in pairs that multiply to the number n: i.e. the pair k and $\frac{n}{k}$. Squares are the exception with an odd number of factors, since the pair of identical numbers $\sqrt{n}$ and $\sqrt{n}$ multiply to n. If a square has 7 factors, the fourth factor happens to be the middle one, $\sqrt{n}$. Thus $n = 8^2 = \boxed{64}$.

5. **Solution 1:** The number of ways to roll the first die is 6, the second die 5 (can't match the first), and the third 4 (can't match the first two), giving $6 \cdot 5 \cdot 4 = 120$ ways. The number of ways to number of ways to roll the first die with no 4's is 5, the second die 4, and the third die 3, for a total of $5 \cdot 4 \cdot 3 = 60$ ways. The desired probability is the complement of the probability that you obtain no 4's (since the numbers are distinct), so the probability is $1 - \frac{60}{120} = \boxed{\frac{1}{2}}$.

   Solution 2: The three die give three distinct numbers. Since there are 6 numbers to choose from, the probability that any number comes up is $\frac{3}{6} = \boxed{\frac{1}{2}}$.

6. Let the two numbers be a and b. Since $a + b = 4x$ and $a - b = 4y$, x and y integers, we get $a = 2(x + y)$ and $b = 2(x - y)$. Thus, a and b are two even numbers that differ by a multiple of 4. There are 6 such pairs $[(0, 4), (0, 8), (4, 8), (2, 6), (2, 10)$, and $(6, 10)]$ with $\binom{11}{2} = 55$ total pairs, so the answer is $\boxed{\frac{6}{55}}$.

7. Each person can have four possible outcomes: HH, HT, TH, and TT, so he flips one head and one tail $\frac{1}{2}$ of the time (call this outcome D) and two heads/two tails $\frac{1}{2}$ of the time (call this outcome S). The three people can have eight outcomes: $DDD, DDS, DSD, SDD, SSD, SDS, DSS$, and SSS. Of these, DDS, DSD, and SDD satisfy the requirements, and the answer is $\boxed{\frac{3}{8}}$.

8. Let C_1 be the scenario that juror 1 decides correctly, I_1 be the scenario that he/she decides incorrectly, and define C_2, I_2, C_3, and I_3 similarly. For the jury to decide correctly, we need either $C_1 \cdot C_2 \cdot I_3, C_1 \cdot I_2 \cdot C_3, I_1 \cdot C_2 \cdot C_3$, or $C_1 \cdot C_2 \cdot C_3$. The corresponding probabilities are $p \cdot p \cdot \frac{1}{2}$, $p \cdot (1-p) \cdot \frac{1}{2}$, $(1-p) \cdot p \cdot \frac{1}{2}$, and $p \cdot p \cdot \frac{1}{2}$. Summing these gives $\boxed{p}$.

9. To roll exactly one six, two of the three dice need to be $1, 2, 3, 4$, or 5 and the other die 6. Since the die with the 6 can be any of the three, the number of ways to do this is $5 \cdot 5 \cdot 3 = 75$; to roll exactly two sixes, one of the three dice need to be 1, 2, 3, 4, or 5 and the other two dice 6. Since the non-6 die can be any of the three, the number of ways to do this is $5 \cdot 3 = 25$; finally, there is only one way the roll three sixes. Since there are $6^3 = 216$ total ways to roll three dice, the expected winning is the probability of each happening weighted by the monetary prize for each. This equals $\frac{75}{216} \cdot \$2 + \frac{25}{216} \cdot \$4 + \frac{1}{216} \cdot \$6 = \boxed{\$1}$.

10. Let p_b be the probability of drawing a quarter at first and p_a be the probability of drawing a quarter at the end, so that $p_b = q$ and $p_a = 1 - q$. Note that the difference between the number of dimes and the number of quarters never changes. Thus, if $p_b > \frac{1}{2}$ then $p_a > \frac{1}{2}$, and if $p_b < \frac{1}{2}$ then $p_a < \frac{1}{2}$. However, since $p_b + p_a = 1$, these scenarios are impossible, so the only possibility is $p_b = q = \boxed{\frac{1}{2}}$.