Johns Hopkins Mathematics Tournament

April 23, 2005

Science Math Solutions: Chemistry of Mountain Dew

(a) Consider n placeholders in a line, one for each object. There are n ways to fill the first position, $n-1$ ways to occupy the second, and so on until there is just one object left for the last spot. Thus, there are $n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 = \boxed{n! \text{ ways}}$ to arrange n different objects in a line.

(b) Imagine taking the line described in the previous question and bending it into a circle. If every object was moved to its adjacent position (in either a clockwise or counterclockwise direction), the circle would be unchanged. There are n such identical configurations (the way it is right now, the arrangement resulting from rotation by one position, that resulting from rotation by two positions, and so on all the way up to rotation by $n-1$ positions), and so we divide the answer to the previous question by n to give $\boxed{(n-1)! \text{ arrangements}}$.

Another way to consider this problem is as follows. Consider n people who want to sit around a circular table. The first person to sit can choose any of the n chairs without his absolute position mattering. It is his position *relative* to the others that defines differences in the arrangement. Now for the chair directly to the right of him, there are $n-1$ choices, and for the position directly to the right of *that* person, there are $n-2$. As we go around the circle this way, there are a total of $(n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1 = (n-1)!$ ways.

(c) The logic from the previous question tells us that absolute position is not important and that we must instead consider relative position of the chemical groups. Thus, place one of the three chemical groups, say the $\mathbf{CO_2H}$ group, in one of the six spots. Number the six positions on the circle (P_1 through P_6) in a clockwise fashion, with P_1 being the position you placed the $\mathbf{CO_2H}$ group. From the picture of the molecule, the **OH** group can be placed at either P_2 or P_6. If **OH** is at P_2, **Br** must be placed at P_5, and if **OH** is at P_6, **Br** must be located at P_3. There are thus 2 acceptable choices for **OH** (of 5 empty positions), and the choice of **OH** forces only 1 acceptable spot for **Br** (of the 4 remaining positions). The probability that three groups randomly placed on the ring yield the desired chemical is thus $\frac{2}{5} \cdot \frac{1}{4} = \boxed{1/10}$.

(d) Inscribe the great circle[1] of the spherical object in the regular hexagon of side a. Perpendicularly bisect one of the sides of the hexagon with a line segment coming from the center of the circle. The length of this perpendicular bisector is the maximum radius r of the sphere. Notice that drawing another line segment from the center of the circle to one of the endpoints of the bisected hexagon side produces a $30^o - 60^o - 90^o$ right triangle with leg lengths $\frac{a}{2}$ and r, and hypotenuse length a. The Pythagorean Theorem yields $r = \boxed{a\sqrt{3}/2}$.

[1]The **great circle** of a sphere is the cross-sectional circle of the sphere that passes through the center of the sphere.

(e) The given fact that $\frac{c}{d} = \frac{e}{f} = K$ implies that $c = dK$ and $e = fK$. The caffeine content K' of the mixture is:

$$K' = \frac{c+e}{d+f} = \frac{dK+fK}{d+f} = \frac{K(d+f)}{d+f} = K.$$

Since the caffeine content is unchanged in the mixture, $\boxed{x = 1}$. Those who know basic chemistry will realize that this is logical, as a mixture of two solutions with an equal concentration of some substance (in this case caffeine) has the same concentration of that substance.

(f) The H-NMR spectrum shows 28 types of hydrogen atoms as there are 28 peaks in the spectrum. Only 6 of these peaks have $ppm > 4.00$ so 6 types of hydrogen atoms are near double bonds. Ignoring the areas under the peak, the probability that a hydrogen atom randomly chosen from the sample is near a double bond is $\frac{6}{28} = \boxed{3/14}$.

When considering the peak dimensions, notice that the peaks with $ppm > 4.00$ are much smaller and contain less area than those with lesser *ppm*. Since the number of hydrogens with a given *ppm* is proportional to the area under that peak, there are actually fewer hydrogen atoms near double bonds when considering and we cannot give as much weight to the double-bond peaks as we give to the non-double-bond peaks. Although we cannot compute a precise probability, we can say with certainty that this exact probability is $\boxed{\text{much less}}$ than that calculated in the first part.

Johns Hopkins Mathematics Tournament

April 23, 2005

Science Math Solutions: Designing Plastic Bags

(a) The square catalyst contains all points (x, y) for which the coordinates satisfy $-5 \le x \le 5$ and $-5 \le y \le 5$. Of these, there are $11 \cdot 11 = 121$ points with integer coordinates, and so $\boxed{121}$ pre-polymers can bind to one side of the catalyst.

(b) Carefully tabulate a list of the number of pre-polymers on the catalyst at a given time. You will notice that there are 2 pre-polymers after 2 minutes, 3 pre-polymers after 6 minutes, 4 pre-polymers after 10 minutes, and in general, α pre-polymers after $\boxed{4(\alpha - 2) + 2}$ minutes. If you answered the first question correctly ($\alpha = 121$ pre-polymers), you would find that the catalyst would be completely loaded after $4(121 - 2) + 2 = \boxed{478 \text{ minutes}}$.

(c) Note that the desired expect mass M is given by:

$$M = 1 \cdot \frac{1}{1} + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + ...$$

Recall that the infinite series $a + ar + ar^2 + ...$ with $-1 \le r \le 1$ has sum $S = \frac{a}{1-r}$. This equation does not directly apply to the given series, but there are several clever methods to evaluate M. We present one solution (not particularly the shortest) of breaking the given series into an infinite number of simple infinite series and *then* applying the infinite sum formula:

$$\begin{aligned}
M &= 1 \cdot \frac{1}{1} + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + ... \\
&= 1 + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + ... \\
&= \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...\right) + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...\right) + \left(\frac{1}{4} + \frac{1}{8} + ...\right) + \left(\frac{1}{8} + ...\right) + ... \\
&= \left(\frac{1}{1 - \frac{1}{2}}\right) + \left(\frac{\frac{1}{2}}{1 - \frac{1}{2}}\right) + \left(\frac{\frac{1}{4}}{1 - \frac{1}{2}}\right) + \left(\frac{\frac{1}{8}}{1 - \frac{1}{2}}\right) + ... \\
&= \frac{1}{1 - \frac{1}{2}} \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...\right) \\
&= \frac{1}{1 - \frac{1}{2}} \cdot \frac{1}{1 - \frac{1}{2}} \\
&= \boxed{4 \text{ kg}}.
\end{aligned}$$

(d) The key to this problem is to note that the number of polymers in the bag N and the stretch limit S never change. We are given that the bag was just about to break when

$T_{bag} = 14^o$C and $M = 2$ kg. Plugging these quantities and the known value $S = 5$ (at the breaking point) into the equation $M = \frac{T_{bag}}{N} S$ yields $N = 35$ polymers.

In the grocery store, we load the bag to $M = 3$ kg, and now that we know both N and S (the constants), we can solve for the temperature T_{bag} at which the bag will break. Using the same formula tells us that the bag will break when $T_{bag} = 21^o$C. We now need to find the time t at which the temperature actually reaches this point:

$$\begin{aligned} T_{bag} &= 24e^{-t} + 4 \\ e^{-t} &= \frac{T_{bag} - 4}{24} \\ t = -\ln\left(\frac{T_{bag} - 4}{24}\right) &= -\ln\left(\frac{21 - 4}{24}\right) \\ t = -\ln\left(\frac{17}{24}\right) &= \boxed{\ln\left(\frac{24}{17}\right) \text{ hours}}. \end{aligned}$$

This time t is approximately 21 minutes, which is plenty of time to get to your car.