Johns Hopkins Mathematics Tournament

April 23, 2005

Algebra Solutions

1. Rewrite the provided expression as

$$a^2 + b^2 - 6a = b^2 + (a-3)^2 - 9.$$

Since the minimum value of a squared term is zero, the answer is $\boxed{-9}$.

2. Let the rectangle's length be L and its width be W. We have $2L + 2W = 25$, so $W = 12.5 - L$. Also, $LW = 25$. Substituting W into the area equation gives $12.5L - L^2 = 25$. Applying the quadratic formula and taking the larger value, we get $L = \boxed{10}$.

3. If the roots are p and q, we have

$$(x-p)(x-q) = x^2 + 2bx + 1 = 0.$$

Expanding the product on the left shows that $p + q = -2b$ and $pq = 1$. Use these values to find an expression for $(p-q)^2$:

$$\begin{aligned}(p-q)^2 &= p^2 - 2pq + q^2 \\ &= p^2 + 2pq + q^2 - 4pq \\ &= (p+q)^2 - 4pq \\ &= (-2b)^2 - 4 = 4b^2 - 4.\end{aligned}$$

The desired difference then follows by taking the square root of both sides.

$$\begin{aligned}p - q &= \sqrt{4b^2 - 4} \\ &= \boxed{2\sqrt{b^2 - 1}}.\end{aligned}$$

4. We will sum each term of $f(x)$ separately. $2^0 + 2^1 + \ldots + 2^8$ is a geometric series, with sum $\frac{1-2^8}{1-2} = 255$; $0 + 1 + \ldots + 8$ is the eighth triangular number $8 \cdot \frac{9}{2} = 36$; finally, we subtract $4 \cdot 9 = 36$, so the total sum is $255 + 36 - 36 = \boxed{255}$.

5. **Solution 1:** $496 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 31$. Thus, all factors are of the form $31^a \cdot 2^b$, with $a = 0$ or 1 and $b = 0$, 1, 2, 3, or 4. The sum of these factors is

$$\begin{aligned}(1+31)(1+2+4+8+16) &= 32 \cdot 31 \\ &= \boxed{992}.\end{aligned}$$

Solution 2: 496 is a perfect number, meaning that its factors (other than itself) sum to itself. Thus we add 496 to itself to find that the total sum is

$$496 + 496 = \boxed{992}.$$

6. Summing the first three equations, we get $9a+2b-5c+2d=6$. Subtracting the last equation yields $a+b+c+d=\boxed{2}$.

7. The area of a equilateral triangle is $\frac{\sqrt{3}\cdot x^2}{4}$ with x being its side length. Thus, since $\frac{\sqrt{3}\cdot x^2}{4}=\sqrt{3}$, $x=2$. The y-coordinate of the third vertex is between the y-coordinates of the other two, 0 and 2, so it is 1. The x-coordinate corresponds to the triangle's height, which equals $2\cdot\frac{area}{side}=\frac{2\cdot\sqrt{3}}{2}=\sqrt{3}$, so the coordinates are $\boxed{(\sqrt{3},1)}$.

8. Note that the sum of two of the three factors is always positive ($2a, 2b$, and $2c$). Also, the differences between two of the three factors is nonzero since a, b, and c are distinct $(\pm(2b-2c), \pm(2c-2a), \pm(2a-2b))$. Therefore, we conclude that the three factors are distinct positive factors of 15. The only triplet that satisfies this is $(1,3,5)$. Thus, $2a, 2b$, and $2c$ are $1+3=4, 1+5=6$, and $3+5=8$ in some order, and $abc=\frac{4}{2}\cdot\frac{6}{2}\cdot\frac{8}{2}=\boxed{24}$.

9. Multiplying by $2ab$ and rearranging gives $(a-2)(b-2)=4$. Thus, $a-2$ and $b-2$ are factors (not necessarily positive) of 4. The possible pairs are $(1,4)$, $(2,2)$, $(4,1)$, $(-1,-4)$, $(-2,-2)$, and $(-4,1)$. $(-2,-2)$ translates into $a=b=0$, an extraneous solution; the other $\boxed{5}$ possibilities are valid pairs.

10. The sum can be rewritten as

$$\begin{aligned}
&= \frac{2^2-1^2}{(1\cdot 2)^2}+\frac{3^2-2^2}{(2\cdot 3)^2}+\frac{4^2-5^2}{(3\cdot 4)^2}+\ldots \\
&= (\frac{1}{1^2}-\frac{1}{2^2})+(\frac{1}{2^2}-\frac{1}{3^2})+(\frac{1}{3^2}-\frac{1}{4^2})+\ldots \\
&= \frac{1}{1^2}-(\frac{1}{2^2}-\frac{1}{2^2})-(\frac{1}{3^2}-\frac{1}{3^2})-(\frac{1}{4^2}-\frac{1}{4^2})+\ldots \\
&= \boxed{1}.
\end{aligned}$$