1. Clyde is making a Pacman sticker to put on his laptop. A Pacman sticker is a circular sticker of radius 3 in with a sector of $120°$ cut out. What is the perimeter of the Pacman sticker in inches?

   Answer: $\mathbf{4\pi + 6}$ in

   Solution: The perimeter of a circle with radius 3 in is $2\pi r = 6\pi$. The sector cut out decreases the perimeter by $\frac{120}{360} = \frac{1}{3}$ of its perimeter and adds in two lines of length 3. Thus, the perimeter of the sticker is $\frac{2}{3}(6\pi) + 2 \cdot 3 = \boxed{4\pi + 6}$.

2. In a right triangle, dropping an altitude to the hypotenuse divides the hypotenuse into two segments of length 2 and 3 respectively. What is the area of the triangle?

   Answer: $\mathbf{\frac{5\sqrt{6}}{2}}$

   Solution: Denote the right triangle ABC with hypotenuse BC. Let D be the intersection of the altitude and BC and let $CD = 2$ and $BD = 3$. Triangle ACD is similar to triangle ABC so $\frac{AC}{CD} = \frac{BC}{AC}$. Thus, $AC = \sqrt{BC \cdot CD} = \sqrt{5 \cdot 2} = \sqrt{10}$. Triangle ABD is similar to triangle ABC so $\frac{AB}{BD} = \frac{BC}{AB}$. Thus, $AB = \sqrt{BC \cdot BD} = \sqrt{5 \cdot 3} = \sqrt{15}$. Therefore, the area of ABC is $\frac{1}{2} \cdot \sqrt{10} \cdot \sqrt{15} = \boxed{\frac{5\sqrt{6}}{2}}$.

3. Consider a triangular pyramid $ABCD$ with equilateral base ABC of side length 1. $AD = BD = CD$ and $\angle ADB = \angle BDC = \angle ADC = 90°$. Find the volume of $ABCD$.

   Answer: $\mathbf{\frac{\sqrt{2}}{24}}$

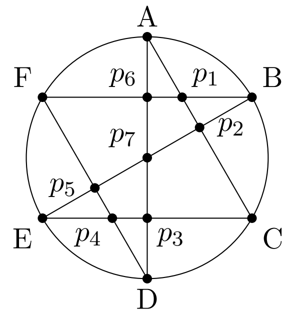
   Solution: Let E be the center of equilateral triangle ABC so that DE is the height of the pyramid. Then $AE = \frac{1}{\sqrt{3}}$. Since $AD = BD$ and $\angle ADB = 90°$, ADB is a 45-45-90 triangle and hence $AD = \frac{AB}{\sqrt{2}} = \frac{1}{\sqrt{2}}$. Thus, by Pythagoras, $DE = \sqrt{AD^2 - AE^2} = \frac{1}{\sqrt{6}}$. Now, the area of the base ABC is $\frac{\sqrt{3}}{4}$ so the volume of $ABCD$ is $\frac{1}{3} \cdot \frac{1}{\sqrt{6}} \cdot \frac{\sqrt{3}}{4} = \boxed{\frac{\sqrt{2}}{24}}$.

4. Suppose you have 15 circles of radius 1. Compute the side length of the smallest equilateral triangle that could possibly contain all the circles, if you are free to arrange them in any shape, provided they don't overlap.

   Answer: $\mathbf{2\sqrt{3} + 8}$

   Solution: We solve the more general question where we wish to pack $\frac{N^2+N}{2}$ circles. The densest packing of a triangular number of congruent circles is to place them in a triangle-like shape, with one in the first row, two in the second, etc. Then, the minimum bounding equilateral triangle will be tangent to all the outer circles, of which there are N per side. The distance between the centers of any two circes is 2. The distance between the center of one of the vertex circles and the closest vertice of the bounding triangle is 2, as can be seen by considering the $30 - 60 - 90$ triangle formed by the radius perpendicular to one of the triangle sides. Then the length of that side from the vertice is $\sqrt{3}$ and so the total length of the side is $2\sqrt{3} + 2(N - 1)$. Since $15 = \frac{5^2+5}{2}$, plugging in $N = 5$ gives us the answer $\boxed{2\sqrt{3} + 8}$.

5. Points $ABCDEF$ are evenly spaced on a unit circle and line segments AD, DF, FB, BE, EC, CA are drawn. The line segments intersect each other at seven points inside the circle. Denote these intersections $p_1, p_2, \ldots, p_7$, where p_7 is the center of the circle. What is the area of the 12-sided shape $Ap_1Bp_2Cp_3Dp_4Ep_5Fp_6$?



Answer: $\frac{5\sqrt{3}}{6}$

Solution 1: To make our calculations cleaner, we will first let the circle have radius 2 and then multiply our answer by $\frac{1}{4}$. If we let the center of the circle be the origin, then $A = (0, 2)$, $B = (\sqrt{3}, 1)$, $C = (\sqrt{3}, -1)$, $D = (0, 2)$, $E = (-\sqrt{3}, -1)$, and $F = (-\sqrt{3}, -1)$. The area of the 12-sided shape is thus twice the sum of the area of right triangles Ap_3C and Bp_1p_2. We easily compute $Ap_3 = 3$ and $p_3C = \sqrt{3}$ so the area of Ap_3C is $\frac{3\sqrt{3}}{2}$. Computing the coordinates of p_1 and p_2, we find that $p_1 = (\frac{1}{\sqrt{3}}, 1)$ and $p_2 = (\frac{\sqrt{3}}{2}, \frac{1}{2})$. Therefore, $Bp_2 = 1$ and $p_1p_2 = \frac{1}{\sqrt{3}}$, so the area of $Bp_1p_2 = \frac{1}{2\sqrt{3}}$. Hence, the area of our 12-sided shape is $\frac{1}{4} \cdot 2 \cdot \left(\frac{3\sqrt{3}}{2} + \frac{1}{2\sqrt{3}}\right) = \boxed{\frac{5\sqrt{3}}{6}}$.

Solution 2: The center is p_7. Because p_7A and p_7C have length 1 we can drop a perpendicular from A to the line through p_7C at a point X. Then AX has length $\sin(\pi/3) = \sqrt{3}/2$. It follows that $\triangle ACp_7$ has area $1/2(\sqrt{3}/2)(1) = \sqrt{3}/4$. Thus the quadrilateral $ACEp_7$ has area $\alpha_1 = \sqrt{3}/2$. Next, notice that BE bisects AC. If the bisection point is Y then YC has length, $(2l)^2 = 2 \cdot 1^2 - 2\cos(2\pi/3) \implies l = \sqrt{3}/2$ Thus Yp_7 has length $\sqrt{1 - (\sqrt{3}/2)^2} = 1/2$. Now call the intersection of AC and BF point Z then the angle Zp_7Y is clearly $\frac{\pi}{6}$ so the area of $\triangle Zp_7Y$ is $\alpha_2 = 1/2(1/2)(1/2\tan(\pi/6)) = \frac{1}{8\sqrt{3}}$. Thus the total area is $2\alpha_1 - 4\alpha_2 = \frac{5\sqrt{3}}{6}$

6. Consider the parallelogram $ABCD$ such that $CD = 8$ and $BC = 14$. The diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E and $AC = 16$. Consider a point F on the segment $\overline{ED}$ with $FD = \frac{\sqrt{66}}{3}$. Compute CF.

Answer: $\sqrt{\frac{148}{3}}$

Solution: By the parallelogram law,

$$(AD)^2 + (BC)^2 + (AB)^2 + (CD)^2 = (AC)^2 + (BD)^2$$
$$14^2 + 14^2 + 8^2 + 8^2 = 16^2 + (BD)^2$$
$$(BD)^2 = 264$$
$$BD = 2\sqrt{66}$$

Thus

$$EF = \frac{2\sqrt{66}}{3}$$

Let $x = CF$.

By Stewart's Theorem:

$$8 \cdot \frac{\sqrt{66}}{3} \cdot 8 + 8 \cdot \frac{2\sqrt{66}}{3} \cdot 8 = x \cdot \sqrt{66} \cdot x + \sqrt{66} \cdot \frac{2\sqrt{66}}{3} \cdot \frac{\sqrt{66}}{3}$$
$$\frac{64\sqrt{66}}{3} + \frac{128\sqrt{66}}{3} = x^2\sqrt{66} + \frac{132\sqrt{66}}{9}$$
$$64\sqrt{66} = x^2\sqrt{66} + \frac{44\sqrt{66}}{3}$$
$$64 = x^2 + \frac{44}{3}$$
$$x^2 = \frac{192-44}{3}$$
$$x = \sqrt{\frac{148}{3}}$$

7. Triangle ABC is isoceles with $AB = AC$. Point D lies on AB such that the inradius of ADC and the inradius of BDC both equal $\frac{3-\sqrt{3}}{2}$. The inradius of ABC equals 1. What is the length of BD?

Answer: $\boldsymbol{\sqrt{3}}$**.**

Solution: It turns out that triangle ABC is equilateral with side length $2\sqrt{3}$ and D is the midpoint of AB. Therefore, $BD = \frac{2\sqrt{3}}{2} = \boxed{\sqrt{3}}$.

8. In a triangle ABC, let D and E trisect BC, so $BD = DE = EC$. Let F be the point on AB such that $\dfrac{AF}{FB} = 2$, and G on AC such that $\dfrac{AG}{GC} = \dfrac{1}{2}$. Let P be the intersection of DG and EF, and X be the intersection of AP and BC. Find $\dfrac{BX}{XC}$.

Answer: $\boldsymbol{\dfrac{2}{3}}$

Solution: Note that DG happens to be parallel to AB as $\frac{BD}{DC} = \frac{AG}{GC} = \frac{1}{2}$. Therefore triangles DEP and BEF are similar so we have $\frac{DP}{BF} = \frac{DE}{BE} = \frac{1}{2}$. This implies that $DP = \frac{BF}{2} = \frac{AB}{6}$. Next, triangles DPX and ABX are similar so we have $\frac{BX}{DX} = \frac{AB}{PD} = 6$. Hence, $BX = \frac{6}{5}BD = \frac{2}{5}BC$ and $XC = BC - BX = \frac{3}{5}BC$. So we conclude that $\frac{BX}{XC} = \boxed{\dfrac{2}{3}}$.

9. In a triangle ABC, two angle trisectors of A intersect with BC at D and E respectively so that B, D, E, C comes in order. If we have $BD = 3$, $DE = 1$ and $EC = 2$, find $\angle DAE$.

Answer: $\boldsymbol{\dfrac{\pi}{4}}$

Solution 1: Let $AB = b$, $AD = d$, $AE = e$, $AC = c$, $\angle DAE = \theta$. Applying the angle bisector formula gives

$$b : e = 3 : 1, \quad d : c = 1 : 2$$

so we have $b = 3e$ and $c = 2d$. Meanwhile, at triangle ABE and ADC the length of angle bisector is given by

$$d = \frac{2be}{b+e}\cos\theta, \quad e = \frac{2dc}{d+c}\cos\theta \qquad ...(*)$$

so we have

$$d = \frac{3e}{2}\cos\theta, \quad e = \frac{4d}{3}\cos\theta.$$

Multiplying those two equations give $\cos^2\theta = 1/2$, so $\theta = \pi/4$.

Formula used for (*) is obtained as follows. In a triangle XYZ let W be the intersection of angle bisector of X and YZ. We can compute the area of XYZ in two ways:

$$\begin{aligned}[XYZ] &= \frac{1}{2}XY \cdot XZ \sin\angle X \\ &= [XYW] + [XWZ] = \frac{1}{2}(XY \cdot XW + XW \cdot XZ)\sin\angle X/2\end{aligned}$$

Solving this for XW and using $\sin\angle X = 2\sin\angle X/2\cos\angle X/2$ gives

$$XW = \frac{2XY \cdot XZ}{XY + XZ}\cos\angle X/2.$$

Solution 2: The locus of points P with $\angle BPD = \angle DPE$ is equivalent to locus of P with $BP : PE = BD : DE = 3 : 1$. This can be found by Apollonian circle theorem, which states that the locus is given by a circle having two points on a line BC as diameter. But as C happens to be on that locus, the circle has DC as diameter. Therefore we have $\angle DAC = 2\angle DAE = \pi/2$, $\angle DAE = \pi/4$.

10. A unit sphere is centered at $(0, 0, 1)$. There is a point light source located at $(1, 0, 4)$ that sends out light uniformly in every direction but is blocked by the sphere. What is the area of the sphere's shadow on the xy plane?

Answer: $\mathbf{6\sqrt{2}\pi}$

Solution: The region in space that is in shadow due to the sphere is a cone. Therefore, the sphere's shadow on the xy plane is the intersection of a cone and a plane, which is an ellipse. We proceed to compute the major and minor axes of the ellipse.

First, note that since the y-coordinate of the sphere's center and the light source both equal 0, one of the axes must lie along the x-axis. The axes of an ellipse are perpendicular to one another, so the remaining axis must be parallel to the y-axis.

Now, consider projecting everything onto the xz plane (that is, simply disregard the y coordinate). The sphere is projected onto a unit circle centerd at $(0, 1)$, the light source is projected to the point $(1, 4)$, and the ellipse is projected onto its horizontal axis. Let ABC be the triangle consisting of the light source A and let B, C be the two ends of the ellipse's axis. The circle is thus the incircle of ABC, and we see that ABC must be a right angle triangle with $\angle ABC = 90°$. Let D be the point where the incircle intersects AB, E be the point where the incircle intersects BC, and F be the point where the incircle intersects AC. Then $AD = AF = 3$, $BD = BE = 1$ and $CF = CE$. By Pythagoras, $AB^2 + BC^2 = AC^2$ so $4^2 + (1 + CE)^2 = (3 + CE)^2$. Solving for CE, we find $CE = 2$, so the horizontal axis of the ellipse $BC = 3$.

Next, we project everything onto the yz plane. This time, the ellipse is projected onto its vertical axis. Again, let A be the light source and B, C be the endpoints of the ellipse's axis. Then ABC is a isoceles triangle with $AB = BC$ and the unit sphere is projected onto the incircle of ABC. If we let D be the intersection of the incircle and AB, E be the intersection of the incircle and AC, and F be the intersection of the incircle and BC, then we have $CE = CF = BD = BF$ and

$AD = AE$. Let O denote the center of the incircle. Then $OA = 3$ and $OD = OE = OF = 1$. By Pythagoras, $AE^2 + OD^2 = OA^2$ so $AE = \sqrt{3^2 - 1^2} = 2\sqrt{2}$. Applying Pythagoras again, to ACF, we have $AF^2 + CF^2 = AC^2$ so $4^2 + CF^2 = (2\sqrt{2} + CF)^2$. Solving for CF, we have $CF = \sqrt{2}$. Thus, the vertical axis BC is equal to $2 \cdot CF = 2\sqrt{2}$.

The sphere's shadow on the xy plane is hence an ellipse with axes 3 and $2\sqrt{2}$ so the area of the shadow is $3 \cdot 2\sqrt{2} \cdot \pi = \boxed{6\sqrt{2}\pi}$.