1. In a gathering of one-eyed, four-legged cyclops and 210 humans, there are 672 legs present. Determine how many eyes there are.

   Answer: 483

   Solution: There are $(672 - 420)/4 = 63$ one-eyed, four-legged cyclops and hence $(210)(2) + (63)(1) = \boxed{483}$ eyes.

2. A triangle with side lengths 6, 10, and 14 has area x. A triangle with side lengths 9, 15, and 21 has area y. Compute $\frac{x}{y}$.

   Answer: $\mathbf{\frac{4}{9}}$.

   Solution: The two triangles are similar with side lengths in the ratio of $\frac{2}{3}$. Therefore, their area is in ratio $\left(\frac{2}{3}\right)^2 = \boxed{\frac{4}{9}}$.

3. Lynnelle really loves peanut butter, but unfortunately she cannot afford to buy her own. Her roommate Jane also likes peanut butter, and Jane buys a new 100mL jar every month. Lynnelle has decided to steal some peanut butter from Jane's jar every day immediately after Jane eats, but to make sure Jane doesn't notice Lynnelle never steals more than 20mL and never steals so much that the amount remaining in the jar is more than halved. For example, if 50mL of peanut butter remains in the jar then Lynnelle will steal 20mL that day (since half of 50mL is 25mL, and Lynelle will steal at most 20mL in one day), and if 8mL remains then Lynnelle will steal 4mL that day (leaving 4mL, half of 8mL). If Jane eats a constant 10mL of peanut butter each day (or the rest of the jar, if the jar has less than 10mL in it) until the jar is empty, compute the amount Lynnelle steals (in mL).

   Answer: 57.5

   Solution: We can step through one day at a time. Before Jane eats any Lynnelle can't steal any. The first day, Jane will eat 10mL, and Lynnelle can then steal 20mL, leaving 70mL. The second day, Jane will eat 10mL, leaving 60, and Lynnelle can steal 20mL, leaving 40. The next day, Jane eats another 10mL leaving 30, and Lynnelle steals 15mL, leaving 15. The next day Jane eats 10mL leaving 5, and Lynnelle steals 2.5mL, leaving 2.5. Finally, the next day Jane will finish the jar. At this point Lynnelle will have stolen $20 + 20 + 15 + 2.5 = \boxed{57.5}$.

4. Consider a unit square $ABCD$. Let E be the midpoint of BC and F the intersection of AC and DE. Compute the area of triangle ADF.

   Answer: $\mathbf{\frac{1}{3}}$

   Solution: The line AC has equation $y = -x + 1$ and DE has equation $y = \frac{1}{2}x$. Therefore AC and DE intersect when $-x + 1 = \frac{1}{2}x$ so when $x = \frac{2}{3}$. Thus, triangle ADF has base $AD = 1$ and height $\frac{2}{3}$ and hence area $\boxed{\frac{1}{3}}$.

5. Tyrant Tal, a super genius, wants to create an army of Tals. He and his clones can clone themselves, but the process takes an entire hour. Once the clone is created, it must wait 2 hours before creating its own clones. So at the end of the first hour, there could be 2 Tals (the original and 1 clone), and a clone created during the fifth hour can clone itself during the eighth hour. Compute the maximum size of his army after 10 hours (Tyrant Tal starts by himself and is a part of his own army).

Answer: 60

Solution:

End of Hour	1	2	3	4	5	6	7	8	9	10
# of Tals	2	3	4	6	9	13	19	28	41	$\boxed{60}$

6. How many ways are there to draw a path between all dots in a 3×3 grid exactly once, where a path from dot to dot may only go horizontally or vertically?

Answer: 20

Solution: There are three basic shapes for such paths, one that looks like an S, one that looks like a 6, and one that looks like a G (with a spur). The S pattern has 4 symmetries (counting rotation and reflection), and the 6 and G patterns have 8 symmetries. The answer is $4 + 8 + 8 = \boxed{20}$.

7. Find the unique $x > 0$ such that $\sqrt{x} + \sqrt{x + \sqrt{x}} = 1$.

Answer: $\mathbf{\frac{1}{9}}$

Solution: We solve

$$\begin{aligned}
\sqrt{x} + \sqrt{x + \sqrt{x}} &= 1 \\
\sqrt{x + \sqrt{x}} &= 1 - \sqrt{x} \\
x + \sqrt{x} &= (1 - \sqrt{x})^2 \\
x + \sqrt{x} &= 1 + x - 2\sqrt{x} \\
3\sqrt{x} &= 1 \\
x &= \boxed{\frac{1}{9}}
\end{aligned}$$

8. Andy has two identical cups, the first one is full of water and the second one is empty. He pours half the water from the first cup into the second, then a third of the water in the second into the first, then a fourth of the water from the first into the second and so on. Compute the fraction of the water is in the first cup right before the 2015th transfer.

Answer: $\mathbf{\frac{1008}{2015}}$

Solution: Insert inductive argument that after an odd number of water transfers, both cups are half filled. Then the 2014th transfer will pour back $\frac{1}{2} \cdot \frac{1}{2015}$ of the water from the second cup into the first, so there is $\boxed{\frac{1008}{2015}}$ of the water in the first cup just before the 2015th transfer.

9. Consider a unit cube and a plane that slices through it. The plane passes through the mid points of two adjacent edges on the top face, two on the bottom face, and the center of the cube. Compute the area of the cross section.

Answer: $\mathbf{\frac{3\sqrt{3}}{4}}$

Solution: Draw the cube to see that the cross-section bust be a regular hexagon. To see this you can note that the by symmetry the plane must intersect the two vertical edges of the cube at their midpoints. This defines all 6 vertices of the hexagon and there remains to draw them to see that it is regular. One side of the hexagon lies in the top face of the cube between the points on the edges that the plane passes through. The side length is thus $\sqrt{2}/2$. The area of the hexagon is thus $6 \times \frac{(\sqrt{2}/2)^2\sqrt{3}}{4} = \boxed{\frac{3\sqrt{3}}{4}}$, 6 times the area of one equilateral triangle of the same side length.

10. Let the sequence a be defined as $a_0 = 2, a_n = 1 + a_0 \cdot a_1 \cdot \ldots \cdot a_{n-1}$.

Calculate $\sum_{i=0}^{2015} \frac{1}{a_i}$. Give your answer in terms of a_{2016}.

Answer: $\boldsymbol{\frac{a_{2016}-2}{a_{2016}-1}}$

Solution: Observe that the sequence follows the recurrence relation $\frac{a_n-2}{a_n-1} + \frac{1}{a_n} = \frac{a_{n+1}-2}{a_{n+1}-1}$. Then $\frac{1}{a_n} = \frac{a_{n+1}-2}{a_{n+1}-1} - \frac{a_n-2}{a_n-1}$. The middle terms in the sum cancel, leaving just $\sum_{i=0}^{2015} \frac{1}{a_i} = \frac{a_{2016}-2}{a_{2016}-1} - \frac{a_0-2}{a_0-1} = \boxed{\frac{a_{2016}-2}{a_{2016}-1}}$.