1. Let $f(x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$. Compute $f'(2)$.

   Answer: 405

   Solution 1: We directly compute $f'(x) = 5x^4 + 20x^3 + 30x^2 + 20x + 5$. Plugging in $x = 2$, we get $\boxed{405}$.

   Solution 2: We first factor $f(x) = (x+1)^5$. Then $f'(x) = 5(x+1)^4$ and plugging in $x = 2$, we get $\boxed{405}$.

2. There are 10 contestants in the Stanford Mountaineering Tournament, numbered 1 to 10. At time t the height of the contestant number n is given by $x_n(t) = t^n$. Compute the average speed of the 10 contestants at time $t = 2$.

   Answer: 921.8

   Solution: The total speed of the 10 contestants is given by

$$\begin{aligned} x_1'(2) + x_2'(2) + \cdots + x_{10}'(2) &= \frac{d}{dt}\left(x_1(t) + \cdots + x_{10}(t)\right)\Big|_{t=2} \\ &= \frac{d}{dt}(t + t^2 + \cdots + t^{10})\Big|_{t=2} \\ &= \frac{d}{dt}\left(\frac{t^{11} - t}{t - 1}\right)\Big|_{t=2} \\ &= \frac{(11t^{10} - 1)(t-1) - (t^{11} - t)}{(t-1)^2}\Big|_{t=2} \\ &= 9218 \end{aligned}$$

   Therefore the average speed is $\frac{9218}{10} = \boxed{921.8}$.

3. Moor is trying to paint the interval $[0, 5]$ using red and green paints. For some reason, painting at the point x using red paint costs $2x$ dollars per unit length and using green paint costs x^2 dollars per length. What is the minimum amount of money Moor needs to spend to paint the entire interval if he's allowed to change colors as he paints?

   Answer: $\mathbf{\frac{71}{3}}$

   Solution: For $0 \le x \le 2$, it is cheaper for Moor to use green paint since $x^2 \le 2x$ in the interval $[0, 2]$. For the interval $[2, 5]$, it is cheaper for Moor to use red paint. Thus, the minimum amount of money Moor needs to spend is

$$\int_0^2 x^2\,dx + \int_2^5 2x\,dx = \boxed{\frac{71}{3}}$$

   .

4. Compute

$$\frac{d}{dx}\prod_{n=1}^{2014}\left(x + \frac{1}{n}\right)\Bigg|_{x=0}$$

   Answer: $\mathbf{\frac{2015}{2\times 2013!}}$

   Solution: Use the product rule,

$$\frac{d}{dx}\prod_{n=1}^{2014}\left(x + \frac{1}{n}\right)\Bigg|_{x=0} = \sum_{k=1}^{n}\prod_{n=1}^{2014}\frac{(x + \frac{1}{n})}{x + \frac{1}{k}}\Bigg|_{x=0} = \sum_{k=1}^{n}\prod_{n=1}^{2014}\frac{\frac{1}{n}}{\frac{1}{k}} = \sum_{k=1}^{n}\frac{k}{2014!} = \frac{2015}{2\times 2013!}$$

5. For some positive pairs of real numbers (α, β) the following limit exists and is nonzero. Compute it in terms of α and β:

$$\lim_{x\to 0} \frac{\sin x^\alpha}{\cos x^\beta - 1}$$

Answer: $\mathbf{-2}$

Solution 1: Since $\alpha, \beta > 0$, we may apply l'Hopital's rule so that the given limit is equal to $-\frac{\alpha}{\beta} \lim_{x\to 0} \frac{x^{\alpha-\beta}\cos x^\alpha}{\sin x^\beta}$. Noting that $\cos(0) = 1$ and that the existence of our limit is assumed, we can simplify the expression to just $-\frac{\alpha}{\beta} \lim_{x\to 0} \frac{x^{\alpha-\beta}}{\sin x^\beta}$. Now we use the fact that the limit must be nonzero: $-\frac{\alpha}{\beta} \lim_{x\to 0} \frac{x^{\alpha-\beta}}{\sin x^\beta} = -\frac{\alpha}{\beta} \lim_{x\to 0} \frac{x^\beta}{\sin x^\beta} \cdot x^{\alpha-2\beta} = -\frac{\alpha}{\beta} \lim_{x\to 0} x^{\alpha-2\beta}$. Then $\alpha = 2\beta$ giving us the answer $\lim_{x\to 0} \frac{\sin x^\alpha}{\cos x^\beta - 1} = \boxed{-2}$.

Solution 2: This problem may easily be solved by using Taylor series expansions, which are very well known for sine and cosine. Specifically,

$$\lim_{x\to 0} \frac{\sin x^\alpha}{\cos x^\beta - 1} = \lim_{x\to 0} \frac{x^\alpha - \frac{x^{3\alpha}}{6} + \dots}{\left(1 - \frac{x^{2\beta}}{2} + \dots\right) - 1} = \lim_{x\to 0} x^{\alpha-2\beta} \left[\frac{1 - \frac{x^{2\alpha}}{6} + \dots}{-\frac{1}{2} + \frac{x^{2\beta}}{24} - \dots}\right].$$

Because the limit is nonzero, we must have $\alpha = 2\beta$, and then plugging $x = 0$ into the last expression yields the answer $\lim_{x\to 0} \frac{\sin x^\alpha}{\cos x^\beta - 1} = \frac{1}{-\frac{1}{2}} = \boxed{-2}$.

6. Compute

$$\int_0^2 \sqrt{(2-x)\left(\sqrt{x} + \sqrt{x+2}\right)^2}\, dx.$$

Answer: $\mathbf{\frac{3\pi}{2}}$

Solution: Distribute into $\sqrt{2-x}\sqrt{x} + \sqrt{2-x}\sqrt{x+2} = \sqrt{2x - x^2} + \sqrt{4 - x^2}$. This is the sum of two circle segments, half of a circle with radius 1 (centered at $(1, 0)$) and a quarter of a circle with radius 2. So the answer is $\frac{1}{2}\pi + \frac{1}{4}4\pi = \boxed{\frac{3\pi}{2}}$.

7. Given that it converges, compute the following infinite product:

$$\prod_{n=1}^{\infty} \frac{5^{2^{-n}} + 3^{2^{-n}}}{2}.$$

Answer: $\mathbf{\frac{2}{\ln\left(\frac{5}{3}\right)}}$

Solution: Let

$$P_n = \prod_{k=1}^{n} \frac{5^{2^{-n}} + 3^{2^{-n}}}{2} = \frac{1}{2^n} \prod_{k=1}^{n} \left(5^{2^{-n}} + 3^{2^{-n}}\right).$$

Observe that $2^n \left(5^{2^{-n}} - 3^{2^{-n}}\right) P_n = 2$ for all n. In order to compute the limit

$$\lim_{n\to\infty} 2^n \left(5^{2^{-n}} - 3^{2^{-n}}\right)$$

we remove the restriction to the naturals and let the limit to infinity be realized along the reals - note, if this limit exists, then the limit of the sequence must also exist and be the same. We compute

$$\lim_{x\to\infty} 2^x\left(5^{2^{-x}}-3^{2^{-x}}\right)=\lim_{x\to\infty}\frac{5^{2^{-x}}-3^{2^{-x}}}{2^{-x}}=\lim_{x\to\infty}\frac{(2^{-x})'\left(\ln(5)5^{2^{-x}}-\ln(3)3^{2^{-x}}\right)}{(2^{-x})'}=$$

$$=\lim_{x\to\infty}\ln(5)5^{2^{-x}}-\ln(3)3^{2^{-x}}=\ln(5)-\ln(3)=\ln\left(\frac{5}{3}\right).$$

Thus,

$$P_n=\frac{2}{2^n\left(5^{2^{-n}}-3^{2^{-n}}\right)}\to\boxed{\frac{2}{\ln\left(\frac{5}{3}\right)}}\text{ as } n\to\infty.$$

8. Compute

$$\frac{1}{\pi}\int_0^\pi\left(\frac{\sin(10x)}{\sin x}\right)^2dx$$

.

Answer: 10

Solution: Let $z=e^{ix}$ so that $\sin x=\frac{z-z^{-1}}{2i}$ and $\sin(10x)=\frac{z^{10}-z^{-10}}{2i}$. We utilize these representations of $\sin x$ and $\sin(10x)$ to simplify $\frac{\sin(10x)}{\sin x}=\frac{z^{10}-z^{-10}}{z-z^{-1}}=z^{-9}\frac{z^{20}-1}{z^2-1}=z^{-9}\frac{(z^2)^{10}-1}{z^2-1}=z^{-9}(z^{18}+z^{16}+\ldots+1)=(z^9+z^7+\ldots+z+z^{-1}+\ldots+z^{-7}+z^{-9})=((z^9+z^{-9})+\ldots+(z+z^{-1}))$. The expression in the integral is the square of our "simplification" but before squaring it, I will note several important things that simplify all of the work. First, when we square it, every power of z will be even. Second of all, $z^{2k}+z^{-2k}=2\cos(2kx)$ has integral 0 over the interval 0 to π. Thirdly, we expand the square $\left(\sum_{k=1}^5(z^{2k-1}+z^{-2k+1})\right)^2=\sum_{k=1}^5(z^{2k-1}+z^{-2k+1})^2+2\sum_{1\le k<j\le 5}(z^{2k-1}+z^{-2k+1})(z^{2j-1}+z^{-2j+1})=\sum_{k=1}^5(2+z^{4k-2}+z^{-4k+2})+2\sum_{1\le k<j\le 5}\left((z^{2(k+j)-2}+z^{-2(k+j)+2})+(z^{2(k-j)-2}+z^{-2(k-j)+2})\right)$. So the integral of all of these sums will be 0, with the exception of $\sum_{k=1}^5 2=10$. So we have that the expression in our problem is equal to $\frac{1}{\pi}\cdot(10\pi)=10$, so $\boxed{10}$ is the answer.

A quick comment. Say you have a collection of raffle tickets with n digits in base m (to be clear, if $n=6, m=10$ the tickets are $000000, 000001, \ldots, 999998, 999999$). Call a ticket "happy" if n is even and the sum of the first $\frac{n}{2}$ digits is equal to the sum of the last $\frac{n}{2}$ digits. Then the number of happy raffle tickets for any given n and m is equal to $\frac{1}{\pi}\int_0^\pi\left(\frac{\sin(mx)}{\sin x}\right)^n dx$. If one somehow knew this ahead of time, then clearly, for $n=2, m=10$ we have $00, 11, \ldots, 99$ are the $\boxed{10}$ happy tickets! Also, it is a fun exercise to prove this formula, and I highly recommend at least trying!

9. Compute:

$$\sum_{k=1}^\infty\frac{(\pi-3)^{2^{k-1}}}{1-(\pi-3)^{2^k}}.$$

Answer: $\frac{\pi-3}{4-\pi}$

Solution: Clearly we should consider the sum, $f(x)=\sum_{k=1}^\infty\frac{x^{2^{k-1}}}{1-x^{2^k}}$, with $x<1$. Aside from being terrible at $x=1$, notice that for $x>1$ this sum need not converge and is not well defined despite the fact that $x \to \infty$ could be well behaved. If we factor the denominator the sum can be written:

$$f(x) = \sum_{k=1}^{\infty} \frac{x^{2^{k-1}}}{1 - x^{2^k}} = \sum_{j=0}^{\infty} \frac{1}{\left(\frac{1}{x}\right)^{2^j} - x^{2^j}} = \sum_{j=0}^{\infty} -\frac{csch(2^j r)}{2}$$

where $e^r = x$. Now we can apply the beautiful reduction formula relating the hyperbolic cotangent and cosecant, $csch(z) = coth(z/2) - coth(z)$. Then the sum telescopes!!!

$$\sum_{j=0}^{\infty} \left(-\frac{coth(2^{j-1} r)}{2} + \frac{coth(2^j r)}{2} \right)$$

To accurately resolve the telescoping we need to be careful about the limit because it is an infinite sum and re-arrangements might not be valid. To be careful we should write:

$$= \lim_{N \to \infty} \sum_{j=0}^{N} \left(-\frac{coth(2^{j-1} r)}{2} + \frac{coth(2^j r)}{2} \right) = \lim_{N \to \infty} \left(-\frac{coth(2^{-1} r)}{2} + \frac{coth(2^N r)}{2} \right)$$

To take the limit recall that $r < 0$, and $coth(-x) = \frac{1+e^{2x}}{1-e^{2x}}$ so the limit is,

$$-\frac{coth(r/2)}{2} - \frac{1}{2} = \frac{x}{1-x}$$

The answer follows.

The fact that the sum cannot be telescoped as an infinite sum can be seen from a more general perspective. It is easy to see that $f(x \to 1^-) = \infty$ so if we expect to compute f on the entire interval $[0, 1)$ then it certainly will not be a uniformly convergent sum and our answer will diverge at 1. This means that we may not be able to re-arrange terms which is what telescoping entails. This is why we must telescope the finite sum first.

10. Consider the real-valued differential equation $u''(x) = u^2(x) - u^5(x)$. Suppose that $u'(0) = 7$ and $u(0) = 2$, compute the largest value of $|u'(x)|$.

Answer: $\frac{14}{\sqrt{3}}$

Solution: First integrate the equation once by multiplying by u':

$$0 = u'(u'' - u^2 + u^5) = \frac{d}{dx} \left(\frac{(u')^2}{2} - \frac{u^3}{3} + \frac{u^6}{6} \right)$$

Now, notice that because the derivative is zero, the quantity in the brackets on the RHS must be constant for all x, call it K. In particular at $x = 0$, we have $K = 49/2 - 2^3/3 + 2^6/6 = 65/2$. Thus, $(u')^2 = 2K + \frac{2u^3}{3} - \frac{u^6}{3}$, and the maximum occurs when $u^2 - u^5 = 0 \implies u = 0$ or $u = 1$. By inspection $u = 1$ is greater. So the maximum is, $\sqrt{65 + 1/3} = \boxed{\frac{14}{\sqrt{3}}}$.

For purposes of rigor we should argue that there exists x such that $u(x) = 0$. Define $v = u^3$ and $w = u'$. Then it is clear that the solutions u satisfy: $w^2 = 65 + 2/3v - v^2/6$ These are ellipses centered in the $w - v$ plane, that clearly intersect the $v = 0$ axis. Now the solutions u must correspond to continuous curve that is a subset of this ellipse. It suffices to show that u covers the whole ellipse for the desired result.

In the new variables the ODE has a phase curves in $\mathbb{R}^2$ according to the equation:

$$(w', u') = (u^2 - u^5, w)$$

By uniqueness of solutions to ODEs the point given in the problem defines a unique phase curve on which the solution lies. It must also satisfy the equation of the ellipse. Finally it covers the entire ellipse because this vector field is bounded away from 0 on the ellipse:

$$(w', u') = \left(u^2 - u^5, \sqrt{2\left(\frac{65}{2} + \frac{u^3}{3} - \frac{u^6}{6}\right)}\right) \neq 0$$

therefore the flow cannot approach a fixed point (or limit point) on the ellipse, so it covers the ellipse.