1. In a Super Smash Brothers tournament, $\frac{1}{2}$ of the contestants play as Fox, $\frac{1}{3}$ of the contestants play as Falco, and $\frac{1}{6}$ of the contestants play as Peach. Given that there were 40 more people who played either Fox or Falco than who played Peach, how many contestants attended the tournament?

   Answer: 60

   Solution: Let x denote the number of contestants in the tournament. Then $\frac{1}{2}x + \frac{1}{3}x - \frac{1}{6}x = 40$. Thus, $\frac{2}{3}x = 40$ and hence $x = \boxed{60}$ contestants attended the tournament.

2. Find all pairs (x, y) that satisfy

$$x^2 + y^2 = 1$$
$$x + 2y = 2$$

   Answer: $(\mathbf{0}, \mathbf{1})$ and $(\frac{\mathbf{4}}{\mathbf{5}}, \frac{\mathbf{3}}{\mathbf{5}})$

   Solution: The second equation tells us that $x = 2-2y$. Substituting this into the first equation, we have

$$\begin{aligned} (2-2y)^2 + y^2 &= 1 \\ 5y^2 - 8y + 3 &= 0 \\ (5y-3)(y-1) &= 0 \end{aligned}$$

   Thus, $y = \frac{3}{5}$ or $y = 1$. Plugging in these values of y and solving for x, we find that the possible solutions to the system are $\boxed{(0,1), \left(\frac{4}{5}, \frac{3}{5}\right)}$.

3. Find the unique $x > 0$ such that $\sqrt{x} + \sqrt{x + \sqrt{x}} = 1$.

   Answer: $\frac{\mathbf{1}}{\mathbf{9}}$

   Solution: We solve

$$\begin{aligned} \sqrt{x} + \sqrt{x + \sqrt{x}} &= 1 \\ \sqrt{x + \sqrt{x}} &= 1 - \sqrt{x} \\ x + \sqrt{x} &= (1 - \sqrt{x})^2 \\ x + \sqrt{x} &= 1 + x - 2\sqrt{x} \\ 3\sqrt{x} &= 1 \\ x &= \boxed{\frac{1}{9}} \end{aligned}$$

4. Find the sum of all real roots of $x^5 + 4x^4 + x^3 - x^2 - 4x - 1$.

   Answer: $-\mathbf{3}$

Solution 1: Notice that $x^5 + 4x^4 + x^3 - x^2 - 4x - 1 = x^3(x^2 + 4x + 1) - (x^2 + 4x + 1) = (x^3 - 1)(x^2 + 4x + 1)$. The only real root of $x^3 - 1$ is 1, and the real roots of $x^2 + 4x + 1$ are $-2 \pm \sqrt{3}$ by the quadratic formula. Thus, the sum of all real roots of the polynomial is $1 + (-2 + \sqrt{3}) + (-2 - \sqrt{3}) = \boxed{-3}$.

Solution 2: By the rational root theorem, we quickly discover that 1 is a root of the quintic polynomial. Factoring $x - 1$ out, we are left with the quartic $x^4 + 5x^3 + 6x^2 + 5x + 1$. This polynomial is symmetric, so we may factor it as $x^2((x + \frac{1}{x})^2 + 5(x + \frac{1}{x}) + 4)$. Thus, it suffices to find the roots of $(x + \frac{1}{x})^2 + 5(x + \frac{1}{x}) + 4$. Writing $y = x + \frac{1}{x}$, we have $y^2 + 5y + 4$ which has roots $y = -1, -4$. Thus, the roots of the quartic are the solutions to the equations $x + \frac{1}{x} = -1$ and $x + \frac{1}{x} = -4$. Solving the first equation, we have

$$\begin{aligned} x + \frac{1}{x} &= -1 \\ x^2 + x + 1 &= 0 \end{aligned}$$

which has no real roots. The second equation gives

$$\begin{aligned} x + \frac{1}{x} &= -4 \\ x^2 + 4x + 1 &= 0 \end{aligned}$$

which has the roots $x = -2 \pm \sqrt{3}$.

Therefore, the sum of all real roots of the polynomial is $1 + (-2 + \sqrt{3}) + (-2 - \sqrt{3}) = \boxed{-3}$.

5. Let a, b, c, d be integers with no common divisor such that

$$\frac{a\sqrt[3]{4} + b\sqrt[3]{2} + c}{d} = \frac{1}{2\sqrt[3]{4} + \sqrt[3]{2} + 1}$$

Compute $a + b + c + d$.

Answer: 26

Solution: If we let a, b, c, d be rational numbers, then the solution is defined up to a scaling factor. Thus, we will first solve for rational a, b, c assuming $d = 1$ and then scale the solution such that a, b, c, d are all integers with no common divisor.

Now, we wish to find rational a, b, c such that $a\sqrt[3]{4} + b\sqrt[3]{2} + c = \frac{1}{2\sqrt[3]{4} + \sqrt[3]{2} + 1}$. Cross multiplying and simplifying, we obtain the equation:

$$(a + b + 2c)\sqrt[3]{4} + (4a + b + c)\sqrt[3]{2} + (2a + 4b + c - 1) = 0$$

Since a, b, c are to be rational, the coefficients of $\sqrt[3]{4}$, $\sqrt[3]{2}$, and the constant term must all be zero. This gives the system of equations:

$$\begin{aligned} a + b + 2c &= 0 \\ 4a + b + c &= 0 \\ 2a + 4b + c &= 1 \end{aligned}$$

Solving this system, we find that $a = -\frac{1}{23}$, $b = \frac{7}{23}$, and $c = -\frac{3}{23}$. This was under the assumption that $d = 1$, so if we scale everything up by 23, we find that $a = -1$, $b = 7$, $c = -3$, and $d = 23$. Therefore, $a + b + c + d = \boxed{26}$.

Solution: There must exist polynomials $p(x) = ax^2 + bx + c$ and $q(x) = dx + e$ so that $p(x)(2x^2 + x + 1) + q(x)(x^3 - 2) = 1$ for all x, as the polynomials are relatively prime. We expand the expression to obtain

$$(2a + d)x^4 + (a + 2b + e)x^3 + (a + b + 2c)x^2 + (b + c - 2d)x + (c - 2e) = 1,$$

where all of these coefficients are equal to 0 except for the last one which is equal to 1. Thus, we have a system of linear equations. However we solve the system, we find that $a = -\frac{1}{23}, b = \frac{7}{23}, c = -\frac{3}{23}$. When we plug in $x = \sqrt[3]{2}$ into the polynomial expression, we get $\frac{1}{2\sqrt[3]{4}+\sqrt[3]{2}+1} = \frac{-\sqrt[3]{4}+7\sqrt[3]{2}-3}{23}$ so $a = -1$, $b = 7$, $c = -3$, and $d = 23$. Therefore, $a + b + c + d = \boxed{26}$.

6. Let $f(a, b) = \frac{1}{a+b}$. Suppose that x, y, z are distinct integers such that $x + y + z = 2015$ and $f(f(x, y), z) = f(x, f(y, z))$. Compute y.

   Answer: -2015

   Solution: Since $f(f(x, y), z) = f(x, f(y, z))$, we have that

$$\frac{1}{\frac{1}{x+y} + z} = \frac{1}{x + \frac{1}{y+z}}$$

   Simplifying, we have that

$$\begin{aligned} x + \frac{1}{y + z} &= \frac{1}{x + y} + z \\ x(y + z)(x + y) + (x + y) &= (y + z) + z(x + y)(y + z) \\ x\left((y + z)(x + y) + 1\right) &= z\left((x + y)(y + z) + 1\right) \\ (x - z)\left((y + z)(x + y) + 1\right) &= 0 \end{aligned}$$

   Since x and z are distinct, $x - z \neq 0$ so we may divide through by $x - z$ to obtain

$$\begin{aligned} (y + z)(x + y) + 1 &= 0 \\ (y + z)(x + y) &= -1 \end{aligned}$$

   Since $x + y + z = 2015$, $y + z = 2015 - x$ and $x + y = 2015 - z$ so

$$(2015 - x)(2015 - z) = -1$$

   Since x, y, z are integers, we have that $x = 2014$ and $z = 2016$ (or the other way around). In either case, $y = 2015 - x - z = 2015 - 2014 - 2016 = \boxed{-2015}$.

7. Compute all pairs (a, b) such that $(x^2 + ax + b)^2 + a(x^2 + ax + b) - b = (x - r)^4$ for some real number r.

   Answer: $(0, 0), (1, -\frac{1}{4})$

   Solution: Let $P(x) = x^2 + ax + b$ and $Q(x) = x^2 + ax - b$. Then we are looking for a, b such that $Q(P(x))$ has only a single real repeated root. The roots of $Q(P(x))$ are the solutions to the equations $P(x) = r_1$ and $P(x) = r_2$ where r_1, r_2 are the roots of Q. Thus, a necessary condition is for $Q(x)$ to have a repeated root, so we require that the discriminant $a^2 + 4b = 0$. Then the repeated root of Q is $r = -\frac{a}{2}$.

Now, we require that $P(x) = -\frac{a}{2}$ have a repeated root, so the discriminant of $P(x) + \frac{a}{2}$ must be zero. Therefore, we require that $a^2 - 4(b + \frac{a}{2}) = 0$. From earlier, we know that $a^2 + 4b = 0$, so we may substitute in $b = -\frac{a^2}{4}$. Hence, we have the equation

$$\begin{aligned} a^2 - 4(-\frac{a^2}{4} + \frac{a}{2}) &= 0 \\ a^2 - a &= 0 \\ a(a-1) &= 0 \end{aligned}$$

So $a = 0, 1$ with corresponding $b = 0, -\frac{1}{4}$. Thus, the desired pairs of a and b are $\boxed{(0,0), \left(1, -\frac{1}{4}\right)}$.

8. Let a, b, c, d satisfy

$$\begin{aligned} ab + cd &= 11 \\ ac + bd &= 13 \\ ad + bc &= 17 \\ abcd &= 30 \end{aligned}$$

Find the greatest possible value of a.

Answer: $\mathbf{\sqrt{30}}$

Solution: Note that ab, cd are roots of the quadratic equation $x^2 - 11x + 30$ because $ab + cd = 11$ and $ab \cdot cd = abcd = 30$. But this clearly has roots $5, 6$, thus $\{ab, cd\} = \{5, 6\}$. Similarly, we must have that

$$\{ab, cd\} = \{5, 6\}, \quad \{ac, bd\} = \{3, 10\}, \quad \{ad, bc\} = \{2, 15\}.$$

But we have that $a^2 = \frac{ab \cdot ac \cdot ad}{abcd}$ is maximized when we maximize ab, ac, ad. Using our previous result, we set $(ab, cd) = (6, 5), (ac, bd) = (10, 3), (ad, bc) = (15, 2)$ and conclude that the maximum of a^2 is $\frac{6 \cdot 10 \cdot 15}{30} = 30$, thus $a \le \sqrt{30}$. Note that $(a, b, c, d) = \left(\sqrt{30}, \sqrt{\frac{6}{5}}, \sqrt{\frac{10}{3}}, \sqrt{\frac{15}{2}}\right)$ satisfies the equation (b^2, c^2, d^2 were found in analogous ways to a^2), thus we have $a = \boxed{\sqrt{30}}$ is the greatest possible value of a.

9. Given that real numbers x, y satisfy the equation $x^4 + x^2y^2 + y^4 = 72$, what is the minimum possible value of $2x^2 + xy + 2y^2$?

Answer: $\mathbf{6\sqrt{6}}$

Solution 1: Let $a = x^2 + xy + y^2$ and $b = x^2 - xy + y^2$. Then $ab = x^4 + x^2y^2 + y^4 = 72$ and the expression we are trying to minimize is $\frac{a+b}{2} + a$. Substituting $b = \frac{72}{a}$, the expression wo want to minimize becomes $\frac{3a}{2} + \frac{36}{a}$. By AM-GM, this is greater than or equal to $2\sqrt{\frac{3a}{2} \cdot \frac{36}{a}} = 2\sqrt{54} = 6\sqrt{6}$. Finally, we show $6\sqrt{6}$ is achievable. In AM-GM, equality is only achieved when $\frac{3a}{2} = \frac{36}{a}$ so $a = 2\sqrt{6}$. If we let $x = \sqrt{2\sqrt{6}}$ and $y = -\sqrt{2\sqrt{6}}$, then $a = 2\sqrt{6}$ so $6\sqrt{6}$ is achievable. Hence, the minimum possible value is $\boxed{6\sqrt{6}}$.

Solution 2: Let $a = x^2 + y^2$ and $b = xy$. Then we are trying to minimize $2a + b$. The expression $x^4 + x^2y^2 + y^4$ can be factored as $(x^2 + y^2)^2 - (xy)^2 = (a - b)(a + b)$. Thus, we have the condition

$(a-b)(a+b)=72$. Now, notice that $|x^2+y^2|>|xy|$ for any choice of x,y, so both $a-b$ and $a+b$ are always positive.

Now, suppose $a-b=r$. Then the condition $(a-b)(a+b)=72$ tells us that $a+b=\frac{72}{r}$. Hence, we can solve the system of equations

$$a-b=r$$
$$a+b=\frac{72}{r}$$

to obtain $a=\frac{36}{r}+\frac{r}{2}$ and $b=\frac{36}{r}-\frac{r}{2}$. The expression we are trying to minimize is thus $2a+b=\frac{108}{r}+\frac{r}{2}$. By AM-GM, this is greater than or equal to $2\sqrt{\frac{108}{r}\cdot\frac{r}{2}}=6\sqrt{6}$.

Finally, we show that this is achievable. Equality holds in AM-GM precisely when $\frac{108}{r}=\frac{r}{2}$ or when $r=6\sqrt{6}$. Then $a=4\sqrt{6}$ and $b=-2\sqrt{6}$. So this is achievable by finding x,y such that $x^2+y^2=4\sqrt{6}$ and $xy=-2\sqrt{6}$. One such x,y is $x=\sqrt{2\sqrt{6}}$ and $y=-\sqrt{2\sqrt{6}}$. Thus, the minimum possible value is indeed $\boxed{6\sqrt{6}}$.

10. Consider a sequence defined recursively by $a_n=1+(a_0+1)(a_1+1)\cdots(a_{n-1}+1)$. Find $-2<a_0<-1$ such that

$$\sum_{n=0}^{2015}\frac{a_n}{a_n^2-1}=-\frac{a_0+4}{a_0^2-1}$$

Answer: $\mathbf{3^{-\frac{1}{2^{2015}-1}}-2}$

Solution: First, notice that for $n>1$,

$$\begin{aligned}a_n&=1+(a_0+1)(a_1+1)\cdots(a_{n-2}+1)(a_{n-1}+1)\\&=1+(a_{n-1}-1)(a_{n-1}+1)\\&=a_{n-1}^2\end{aligned}$$

and that $a_1=a_0+2$. Therefore, for $n\geq 1$, $a_n=(a_0+2)^{2^{n-1}}$.

Next, notice that for $n>0$

$$\begin{aligned}\frac{1}{a_n-1}-\frac{1}{a_{n+1}-1}&=\frac{1}{(a_0+1)(a_1+1)\cdots(a_{n-1}+1)}-\frac{1}{(a_0+1)(a_1+1)\cdots(a_n+1)}\\&=\frac{a_n}{(a_0+1)\cdots(a_n+1)}\\&=\frac{a_n}{(a_n-1)(a_n+1)}\\&=\frac{a_n}{a_n^2-1}\end{aligned}$$

Therefore, our sum telescopes as follows

$$\begin{aligned}\frac{a_0}{a_0^2-1}+\sum_{n=1}^{2015}\frac{a_n}{a_n^2-1}&=\frac{a_0}{a_0^2-1}+\sum_{n=1}^{2015}\left(\frac{1}{a_n-1}-\frac{1}{a_{n+1}-1}\right)\\&=\frac{a_0}{a_0^2-1}+\frac{1}{a_1-1}-\frac{1}{a_{2016}-1}\end{aligned}$$

Using the fact that $a_1 = a_0 + 2$ and $a_{2016} = (a_0+2)^{2^{2015}}$, we can simplify the above expression to

$$\frac{a_0}{a_0^2 - 1} + \frac{1}{a_0+1} - \frac{1}{(a_0+2)^{2^{2015}} - 1}$$

If we let $x = a_0 + 2$, then the condition in the problem statement becomes

$$\begin{aligned}
\frac{x-2}{(x-2)^2-1} + \frac{1}{x-1} - \frac{1}{x^{2^{2015}}-1} &= -\frac{x+2}{(x-2)^2-1} \\
\frac{2x}{(x-1)(x-3)} + \frac{1}{x-1} - \frac{1}{x^{2^{2015}}-1} &= 0 \\
\frac{3x-3}{(x-1)(x-3)} - \frac{1}{x^{2^{2015}}-1} &= 0 \\
\frac{3}{x-3} - \frac{1}{x^{2^{2015}}-1} &= 0 \\
3(x^{2^{2015}} - 1) - (x-3) &= 0 \\
3x^{2^{2015}} - x &= 0
\end{aligned}$$

Since we want $-2 < a_0$, $0 < x$, so we can divide through by x to get

$$\begin{aligned}
3x^{2^{2015}-1} - 1 &= 0 \\
x &= 3^{-\frac{1}{2^{2015}-1}}
\end{aligned}$$

And hence $a_0 = \boxed{3^{-\frac{1}{2^{2015}-1}} - 2}$.