# Johns Hopkins Math Tournament 2019

# Individual Round: Probability and Combinatorics

*February 9, 2019*

## Instructions

- **DO NOT TURN OVER THIS PAPER UNTIL TOLD TO DO SO.**
- This test contains 10 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Only answers written on the appropriate area on the answer sheet will be considered for grading.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, immediately tell your proctor and if necessary, report the error to the front desk after the end of your exam.
- Good luck!

1. In how many ways can four men and four women sit around a circular table such that no two men nor no two women are sitting next to one another? Assume that the seats are indistinguishable, meaning that rotations of a permutation are considered to be equivalent.

2. Alexa performs an experiment as follows: First, she rolls a fair six-sided die. The number she gets on the die determines the number of times that she will flip a coin. She then records the number of heads yielded from these coin flips. When Jose walks in the room where she's conducting the experiment, he overhears that she recorded two heads on a particular trial of her experiment. The probability that she rolled a three in the same trial of her experiment can be expressed in the form $\frac{p}{q}$ with $p, q$ coprime. Find $p + q$.

3. Let $D(n)$ for an integer $n$ denote the largest divisor (possibly 1) of $n$ which is not $n$ itself. Carly chooses an integer $n$ uniformly at random between 1 and 2019 inclusive. The probability that $D(n) > D(2019)$ can be expressed in the form $\frac{m}{n}$ with $m, n$ coprime. Find $m + n$.

4. Suppose a frog starts at position zero on a number line. After each second, the frog will jump either right or left by 1 unit with 50 percent probability each. The probability that the frog will reach position -3 before it reaches position 5 can be expressed in the form $\frac{m}{n}$ where $m, n$ are coprime. Compute $m+n$.

5. Consider a group of five students, each of whom choose a day of the week (including weekends) uniformly at random to meet with their professor. The expected number of days where the professor must meet with at least one student can be expressed in the form $\frac{m}{n}$ with $m, n$ coprime. Compute $m + n$

6. Define a random sequence as follows: First, let $a_0 = 100$. Then, to compute $a_n$, we have that $a_n = a_{n-1}/2$ with 25 percent probability and $a_n = 2a_{n-1}$ with 75 percent probability. The sequence terminates when $a_i < 1$ for some $i$. The probability that this is a finite sequence can be expressed in the form $\frac{m}{n}$ with $m, n$ coprime. Compute $m + n$.

7. An ant is placed on one vertex of a cube. On each move, the ant moves to a random adjacent vertex uniformly at random. What is the expected number of moves until the ant reaches the opposite vertex of the cube (i.e. the vertex which cannot be reached in fewer than three moves)?

8. Samantha starts out with the number $a_1 = 2019$ and generates a sequence of decreasing positive integers. If Samantha currently has $n$ terms $a_1, a_2, \ldots,$ and $a_n$ in her sequence, then she selects the next term $a_{n+1}$ uniformly at random from all positive integers less than $a_n$. She repeats this process until she adds the number 1 to her sequence. The probability that the sum of all terms in Samantha's sequence is odd can be expressed as a common fraction $\frac{p}{q}$. What is $p + q$?

9. How many unique ways are there to color the faces of a cube using three colors of paint? The faces of the cube are indistinguishable, so rotating the cube does not create new colorings.

10. In the picture below, the straight lines are called *edges*. A subset of the edges is called a *cycle* if following the edges creates a closed loop. How many unique cycles are contained in the picture below, which contains 7 vertices on each side?