Johns Hopkins Math Tournament 2019

Proof Round: Point Set Topology

February 9, 2019

Problem	Points	Score
1	3	
2	6	
3	6	
4	6	
5	10	
6	6	
7	8	
8	6	
9	8	
10	8	
11	9	
12	10	
13	14	
Total	100	

Instructions

- The exam is worth 100 points; each part's point value is given in brackets next to the part.
- To receive full credit, the presentation must be legible, orderly, clear, and concise.
- If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says justify or prove, then you must prove your answer rigorously.
- Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa. There is an exception for problems in two parts (for example 5a and 5b). You will not receive any credit for part b if your proof for part a is not correct.
- Pages submitted for credit should be **numbered in consecutive order at the top of each page** in what your team considers to be proper sequential order.
- **Please write on only one side of the answer papers.**
- Put the **team number** (NOT the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

Introduction

In this proof round, we will be introducing some of the basic notions of **topological spaces**. A topological space is just a set with additional structure, called a topology, that gives the set extra mathematical structure. Consider the real numbers $\mathbb{R}$. To this set of real numbers we can impose additional structure by defining the notion of distance of two real numbers. If you have taken calculus, you know that this notion of distance is central to the definitions of convergence, continuity and other important concepts. For example, we can intuitively say a sequence $\{a_n\}$ converges to a point x if the distance between $\{a_n\}$ and x gets closer and closer to 0 as n approaches infinity. An arbitrary topological space is just a generalization of this principle. A topological space is a set imposed with a topology, which is a collection of subsets, called open sets, that satisfy certain properties. These open sets allow us to similarly define a notion of convergence, as we will soon see.

1 Definition of Topological Space

Lets define a topological space formally:

Definition 1.1. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X such that the following properties hold:

- $\emptyset$ and X are in $\mathcal{T}$
- The union of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$
- The intersection of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$

A set X equipped with a topology $\mathcal{T}$ is called a **topological space**. The elements of $\mathcal{T}$ are called **open sets** of X.

Example 1.1. *The three element set $\{x, y, z\}$ with the topology $\mathcal{T} = \{\emptyset, \{x\}, \{y, z\}, \{x, y, z\}\}$ is an example of a topological space.*

Example 1.2. *For any set X, the topology $\mathcal{T} = \{\emptyset, X\}$ gives X the structure of a topological space. This is called the **trivial topology** of X Similarly, the topology defined as the collection of all subsets of X also gives X the structure of a topological space. This is called the **discrete topology** of X*

Problem 1: (3 points) For the three element set $\{x, y, z\}$, does $\mathcal{T} = \{\emptyset, \{x, y\}, \{y, z\}, \{x, y, z\}\}$ define a topology on $\{x, y, z\}$? Justify your answer.

No. Note that the intersection of $\{x, y\}$ and $\{y, z\}$ is $\{y\}$, but $\{y\} \notin \mathcal{T}$.

2 Basis for a Topology

Defining the entire collection of open sets for $\mathcal{T}$ can be very annoying. It is often times easier to define a topology if we specify a generating set,that is, a subset of the topology so that every element of the topology can be specified in terms of this generating set. This generating set is called a basis for the topology. We formalize our discussion with the below definition:

Definition 2.1. Let X be a set and $\mathcal{T}$ a topology for that set. A subset $\mathcal{B}$ of $\mathcal{T}$ is called a **basis** for $\mathcal{T}$ if every element in $\mathcal{T}$ can be written as an arbitrary union of elements of $\mathcal{B}$.

Example 2.1. *For a topological space X with the discrete topology, the collection of one point subsets of X is an example of a basis.*

Definition 2.2. To the real numbers $\mathbb{R}$, we can define a topology whose basis is the collection of open intervals (a, b), a, $b \in \mathbb{R}$. This is called the **standard basis** of $\mathbb{R}$. Note that $\mathbb{R}$ itself can be written as unions of elements of this basis, since for any $x \in \mathbb{R}$ there exist an open interval containing x.

Problem 2: (6 points) Let X be a topological space, $\mathcal{T}$ a topology for X, and $\mathcal{B}$ a basis for $\mathcal{T}$. Show that $\mathcal{B}$ satisfies the following two properties:

1. For each $x \in X$, there exist some $B_i \in \mathcal{B}$ so that $x \in B_i$.
2. For any x belonging to intersection of two basis elements $B_1, B_2 \in \mathcal{B}$, there exist basis element $B_3 \in \mathcal{B}$ so that $x \in B_3 \subseteq B_1 \cap B_2$.

Statement 1 is true since X can be written as union of elements of $\mathcal{B}$ by definition. Statement 2 is also true since $B_1 \cap B_2$ is open and thus can be written as union of elements of $\mathcal{B}$.

Problem 3: (6 points)Let X be a topological space with topology $\mathcal{T}$ as before. Show that if $\mathcal{B}$ is a subset of $\mathcal{T}$ that satisfies statements 1 and 2 in problem 2, then $\mathcal{B}$ is a basis of $\mathcal{T}$.

We wish to show that $\mathcal{T}$ is equal to the collection of all unions of elements of $\mathcal{B}$. Note that every element of $\mathcal{B}$ and hence arbitrary unions of elements of $\mathcal{B}$ are in $\mathcal{T}$. Conversely, given an element U of $\mathcal{T}$, for each $x \in U$ there exist basis element $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U$. Then $U = \bigcup B_x$, so U is an arbitrary union of elements of $\mathcal{B}$ as desired.

Problems 2 and 3 suggest an alternative definition for the definition of a basis. Indeed, a basis for a topological space with topology $\mathcal{T}$ is just a subset of $\mathcal{T}$ that satisfies properties 1 and 2 in problem 2. This alternative definition however, is often times easier to use when determining whether a subset of $\mathcal{T}$ is a basis.

3 Closed Sets

Definition 3.1. A subset A of a topological space X is **closed** if $X \setminus A$ is open.

Example 3.1. *Given the topological space $\mathbb{R}$ with its standard basis, closed intervals $[a, b]$ $a, b \in \mathbb{R}$ are examples of closed sets as $(-\infty, a) \cup (b, \infty)$ is open.*

Example 3.2. *Note that a subset of a topological space can be both open and closed! For example, let X be a topological space with the discrete topology. Then every subset of X is open and closed.*

We now define an important construction in topology. For any subset A of a topological space X we can define the small closed subset of X containing A, called the closure of A, as follows:

Definition 3.2. Given a subset A of a topological space X, the closure of A, denoted as $\overline{A}$, is the intersection of all closed sets containing A.

Note in particular that $A \subseteq \overline{A}$, and that A is closed if and only if $A = \overline{A}$.

The above definition is often times very difficult to use practically when finding the closure of certain sets. Let us now provide a different characterization of the closure:

Definition 3.3. Let X be a topological space and let $x \in X$. Then a **neighborhood** of x is an open subset of X containing x.

Definition 3.4. We often times say that a set A **intersects** a set B if $A \cap B$ is nonempty.

Problem 4: (6 points) Show that if A is a subset of topological space X, then $x \in \overline{A}$ *if and only if* every neighborhood of x intersects A.

Suppose $x \notin \overline{A}$. Then $X \setminus \overline{A}$ is a neighborhood of x that does not intersect A. Conversely, if U is a neighborhood of x that does not intersect A, then $X \setminus U$ is a closed set that contains A and thus $\overline{A}$, and hence $x \notin \overline{A}$ as desired.

Example 3.3. *Let $\mathbb{R}$ be a topological space with its standard topology. Then the closure of $[1, 2)$ in $\mathbb{R}$ is simply $[1, 2]$ as expected, since every neighborhood of 2 intersects $\mathbb{R}$, and for every $x \in \mathbb{R} \setminus [1, 2]$ there exist a neighborhood disjoint from $\mathbb{R}$.*

4 Continuous Functions

Now that we have defined some basic notions of topological spaces, let us now consider functions between topological spaces. One class of functions important in topology are called continuous functions, as these maps preserve many topological properties.

Definition 4.1. Let X and Y be two topological spaces. Then a function $f : X \longrightarrow Y$ is **continuous** if for every open set V in Y, the set $f^{-1}(V)$ is open in X.

The next theorem describes some interesting properties of topological spaces. Note that by the below theorem the closure is preserved under continuous maps.

Theorem 4.1. *Let X and Y be two topological spaces. and let $f : X \longrightarrow Y$ be a continuous map. Then*

1. *$f^{-1}(V)$ is closed in X for any closed set V in Y.*

2. *$f(\overline{A}) \subseteq \overline{f(A)}$ for any subset A in X.*

3. *Let $x \in X$. Then given any neighborhood V of $f(x)$ there exist a neighborhood U of x where $f(U) \subseteq V$.*

Proof: (1) If V is closed in Y, then $Y \setminus V$ is open in Y, and thus $f^{-1}(Y \setminus V)$ is open in X by the definition of a continuous map. Since $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$, $f^{-1}(V)$ is closed as desired.

Problem 5: (10 points) Now prove the second and third part of **Theorem 4.1**.

(2): Let $x \in \overline{A}$. Then given a neighborhood V of $f(x)$, we have that $f^{-1}(V)$ is a neighborhood of x, and thus $f^{-1}(V)$ must intersect A at a point y by the statement in problem 4. Then V intersects $f(A)$ at $f(y)$, and thus $f(x) \in \overline{f(A)}$ as desired, again using the statement in problem 4.
(3): Letting $U = f^{-1}(V)$ does the trick.

Problem 6: (6 points) Note that the definition of continuity does not imply that $f(U)$ is open whenever U is open! Indeed, give an example of a continuous map $f : X \longrightarrow Y$ of topological spaces where U is open in X but that $f(U)$ is not open in Y and justify your answer.

Let $f : X \longrightarrow Y$ be the identity map where X is $\mathbb{R}$ is the discrete topology, and Y is $\mathbb{R}$ with the standard topology. Then $f([0,1]) = [0,1]$, but $[0,1]$ is open in X but not in Y.

Remark: Of course, if f is continuous and f^{-1} exists and is also continuous, then $f(U)$ is open if and only if U is open. Note that then "openness" is preserved under such maps. Since topologies are defined in terms of open sets, these maps, called **homeomorphisms** in topology, preserve all topological properties of a given space. These maps are not surprisingly also called **topological isomorphisms**, where "iso" mean same and "morphism" means shape.

5 Hausdorff Spaces

Now that we have learned some basic notions of point set topology, let us now apply them in a class of topological spaces called Hausdorff spaces. Before we define these spaces, let us first understand the notion of convergence in topology.

Intuitively, a sequence $\{a_n\}$ in $\mathbb{R}$ converges to a point x if the terms a_i in the sequence gets closer and closer to x. Another way to think about this is that given *any* positive real number d, *all but finitely many* of the a_i have distances (from x) less than d. In an arbitrary topological space, in the absence of a distance function, we can define a similar notion of convergence using the concept of neighborhoods.

Definition 5.1. Given a topological space X, we say that a sequence $\{a_n\}$ in X converges to the point x if given *any* neighborhood U of x, there exist a positive integer N so that $x_n \in U$ for all $n \geq N$.

Convergence of sequences is also preserved under continuity:

Problem 7: (8 points) Let $f : X \longrightarrow Y$ be a continuous map of topological spaces. Show that if $\{a_n\}$ is a sequence in X that converges to x, then $\{f(a_n)\}$ is a sequence in Y that converges to $f(x)$.

Let V be any neighborhood of $f(x)$. Then $f^{-1}(V)$ is a a neighborhood of x. By the definition of convergence, all but finitely many of the a_i are in $f^{-1}(V)$. Thus all but finitely many of the $f(a_i)$ are in V. Thus $\{f(a_n)\}$ is a sequence in Y that converges to $f(x)$ as desired.

Unfortunately, in an arbitrary topological space, convergence of sequences can behave very strangely. For example, a sequence can converge to more than one point.

Problem 8: (6 points) For the three element set $\{x, y, z\}$ with the topology $\mathcal{T} = \{\emptyset, \{x, y\}, \{y, z\}, \{y\}, \{x, y, z\}\}$, show that the sequence $y, y, y...$ converges to x, y and z!

$y, y, y...$ converges to x since the neighborhoods of x are $\{x, y\}$ and $\{x, y, z\}$, and y is an element in both sets. Similarly, $y, y, y...$ converges to z as the neighborhoods of z are $\{y, z\}$ and $\{x, y, z\}$, and y is an element in both sets. Finally, $y, y, y...$ converges to y as $\{y\}$ is an open set.

For Hausdorff spaces however, we eliminate this exotic behavior, ensuring that any sequence converges to at most one point!

Definition 5.2. A topological space X is **Hausdorff** if for every pair of distinct points in X, there exist disjoint neighborhoods between these two points (Two sets A, B are disjoint if their intersection is empty).

Problem 9: (8 points) Show that if a topological space X is Hausdorff, then every sequence in X converges to at most one point in X.

Let $\{a_n\}$ be a sequence that converges to x in X. If $\{a_n\}$ also converges to y in X where $y \neq x$, then there exist disjoint neighborhoods U, V of x, y respectively. By the definition of convergence U contains all but finitely many of the a_i. But that means V contains only finitely many of the a_i, a contradiction as desired.

Problem 10:

(a) (4 points) Show that $\mathbb{R}$ with the standard topology is Hausdorff (not surprisingly).

(b) (4 points) Now consider $\mathbb{R}$ with the topology $\mathcal{T} = \{(-n, n) : n \in \mathbb{Z}, n \geq 1\}$. It is trivial to show that $\mathcal{T}$ is a topology. Is $\mathbb{R}$ with this topology Hausdorff? Justify your answer.

(a): Let $x, y \in \mathbb{R}$ be distinct points in $\mathbb{R}$. WLOG let $x < y$. Then let ϵ be a positive real number less than $|x - y|/2$. Then $(x - \epsilon, x + \epsilon)$ and $(y - \epsilon, y + \epsilon)$ are disjoint neighborhoods of $\mathbb{R}$ as desired.

(b): No. The sequence $0, 0, 0...$ for example converges to every real number in $\mathbb{R}$.

Here is a important example of a topological space that is not Hausdorff:

Problem 11: (9 points) Let X be any infinite set. We can define a topology on X, called the **cofinite topology**, as the the collection of all subsets U of X such that $X \setminus U$ is either finite (as a set) or all of X. Show that the cofinite topology does indeed define a topology on X but that X is not Hausdorff.

Let $\mathcal{T}$ be the cofinite topology of X. Note that $X \setminus X$ is finite and that $X \setminus \emptyset$ is all of X, so X, $\emptyset$ are elements of $\mathcal{T}$. Let $T_i \in \mathcal{T}$, then $X \setminus \bigcup T_i = \bigcap (X \setminus T_i)$, and since $X \setminus T_i$ is finite, $\bigcap (X \setminus T_i)$ is finite, so $\bigcup T_i \in \mathcal{T}$. Finally, $X \setminus \bigcap_{1 \leq i \leq n} T_i = \bigcup_{1 \leq i \leq n} (X \setminus T_i)$, which is a finite union of finite sets, and is thus finite, so $\bigcap_{1 \leq i \leq n} T_i \in \mathcal{T}$. Thus the cofinite topology is indeed a topology as desired.

X however is not Hausdorff. In fact, no two open sets of X are disjoint. Indeed, suppose that U, V are disjoint open sets of X. Then $U \subseteq X \setminus V$. But since $X \setminus V$ is finite, $X \setminus U$ is infinite, a contradiction as desired.

Problem 12: (10 points) Show that if a topological space X is Hausdorff, then every finite subset of X is closed. Is the converse true? Justify your answer.

Let $\{x\}$ be a singleton set in X. We show that $\{x\}$ is closed. Indeed, let y be another point different from x, and by the Hausdorff definition there exist disjoint neighborhoods U, V of x, y respectively. V does not intersect $\{x\}$, and thus $y \notin \overline{\{x\}}$. Thus $\overline{\{x\}} = \{x\}$, so $\{x\}$ is closed. Since any finite set is a finite union of singleton sets, finite sets are also closed.

The second part of the question is no. Again consider the natural numbers with the cofinite topology. Every finite set is closed by definition. However, as we have seen already, it is not Hausdorff.

There are many properties when considering continuous functions of Hausdorff spaces. We list two below as an exercise:

Problem 13: (14 points)

(a) We say that an open set U of a topological space X is dense if $\overline{U} = X$. Let $f, g : X \longrightarrow Y$ be two continuous maps of topological spaces where Y is Hausdorff. Show that if there exist a dense set $U \subseteq X$ such that $f(x) = g(x)$ for all $x \in U$, then $f(x) = g(x)$ for all $x \in X$.

(b) Let $f, g : X \longrightarrow Y$ be two continuous maps of topological spaces where Y is Hausdorff. Show that the set $\{x \in X : f(x) = g(x)\}$ is closed in X.

(a): Suppose that $f(x) \neq g(x)$ for some $x \in X$. Then there exist disjoint neighborhoods V, W of $f(x), g(x)$ respectively. Note that then $f^{-1}(V) \cap g^{-1}(W)$ is a neighborhood of x. Since U is dense, it has nonempty intersection with any open set (One can easily see this since the complement of $\overline{U}$, the intersection of closed sets containing U, is just the union of all open sets disjoint from U. Since $\overline{U} = X$, its complement is the null set).

Thus there exist $z \in f^{-1}(V) \cap g^{-1}(W) \cap A$. But then $f(z) = g(z)$, a contradiction as V, W are disjoint, as desired.

(b): Let $\Delta = \{x \in X : f(x) = g(x)\}$. For any element $x \notin X \setminus \Delta$, we see that $f(x) \neq g(x)$ and thus because Y is Hausdorff there exist disjoint neighborhoods U, V of $f(x), g(x)$ respectively. Note that then $W = f^{-1}(U) \cap g^{-1}(V)$ is a neighborhood of x. W is disjoint from Δ since if $x \in W \cap \Delta$, then $f(x) = g(x) \in V \cap W$, a contradiction. Thus by the definition of closure $x \notin \overline{\Delta}$ for any $x \notin \Delta$, so $\overline{\Delta} = \Delta$ as desired.