# Johns Hopkins Math Tournament 2018

# Proof Round: Sequences

*February 17, 2018*

| Section | Total Points | Score |
|---|---|---|
| 1 | 5 | |
| 2 | 20 | |
| 3 | 15 | |
| 4 | 25 | |

## Instructions

- The exam is worth 60 points; each part's point value is given in brackets next to the part.
- To receive full credit, the presentation must be legible, orderly, clear, and concise.
- If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says justify or prove, then you must prove your answer rigorously.
- Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa. There is an exception for problems in two parts (for example 5a and 5b). You will not receive any credit for part b if your proof for part a is not correct.
- Pages submitted for credit should be **numbered in consecutive order at the top of each page** in what your team considers to be proper sequential order.
- **Please write on only one side of the answer papers.**
- Put the **team number** (NOT the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

# Introduction

In this proof round, you will explore the idea of **non-unique factorizations**. In your studies you have probably come across the Fundamental Theorem of Arithmetic, which says every integer greater than 1 is the product of prime numbers, and this factoring is unique (disregarding order of the factors). However, we'll be exploring the cases where unique factorization fails.

For this exam we will adopt the convention that $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ represents the set of integers, $\mathbb{N} = \{1, 2, 3, \ldots\}$ represents the set of natural numbers, and $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ represents the set of natural numbers and zero, and $\mathbb{Q}$ represents the set of rational numbers.

# 1 Preliminaries

## 1.1 Modular Arithmetic

Let $a$, $b$, $c$ be integers. We say that "$a$ is congruent to $b$ modulo $c$" if $a = b + kc$ for some other integer $k \in \mathbb{Z}$. This is written notationally as

$$a \equiv b \pmod{c}.$$

**Example 1**

$$\begin{aligned} 9 &\equiv 0 \pmod{3} \\ -12 &\equiv 10 \pmod{11}. \end{aligned}$$

We say that an integer $x$ **divides** another integer $y$ in a set $S \subseteq \mathbb{Z}$ if there exists an integer $k \in S$ such that $y = kx$. We write

$$x|y$$

to mean $x$ divides $y$. We usually omit the set when it is clear where this division is taking place. For example we could write

$$3|9$$

where it is understood that division is taking place in the natural numbers or the integers.

**Problem 1:** (2 points) For some elements $x$, $y$ of a subset $S \subseteq \mathbb{Z}$, we have that $x|y$ in $S$ implies that $x|y$ in $\mathbb{Z}$. Prove that the converse need not be true by providing a counterexample.
Consider the arithmetic sequence

$$1, 5, 9, 13, 17, 21, 25, \ldots$$

We could see that a natural number $n$ is a member of this sequence if and only if

$$n \equiv 1 \pmod 4$$

We call the set of numbers corresponding to this sequence $\mathbf{H}$ and write

$$\mathbf{H} = \{1 + 4k : k \in \mathbb{N}_0\} = 1 + 4\mathbb{N}_0$$

### 1.2 Monoids

A set $S$ of natural numbers is called **multiplicatively closed** if for every pair of numbers $x, y \in S$, we have that $x \cdot y \in S$.

**Definition 1** *A set $S \subseteq \mathbb{Z}$ is called a* ***multiplicative monoid*** *if $1 \in S$ and $S$ is multiplicatively closed.*

Let $\mathbf{M} = \{4 + 6k : k \in \mathbb{N}_0\} \cup \{1\}$.

**Problem 2:** (3 points) Prove that $\mathbf{H}$ and $\mathbf{M}$ are multiplicative monoids.

## 2 Primes and Irreducibles

For a minute, forget the definition you have in your head about prime numbers.

**Definition 2** *Let $S$ be a multiplicative monoid. We say that an element $p \in S$ is* ***prime*** *if $p|xy$ implies that either $p|x$ or $p|y$, where $x, y \in S$ (note that division is taking place in $S$, not in the integers).*

**Problem 3:** (9 points) Consider the numbers 5, 9, and 33. For each of these elements, prove or disprove that it is prime in $\mathbf{H}$.

**Problem 4:** (9 points) Consider the numbers 4, 10, and 22. For each of these elements, prove or disprove that it is prime in $\mathbf{M}$.

**Definition 3** *Let $S$ be a multiplicative monoid. We say that $x \in S$ is **irreducible** if, for every factorization $x = ab$, where $a, b \in S$, we have that at least one of $a$ or $b$ is equal to $\pm 1$.*

Note that in $\mathbb{Z}$, the definitions of prime and irreducible are the same (you don't have to prove this, but it is worth thinking about why this is true). In an arbitrary multiplicative monoid, however, this need not be the case.

**Problem 5:** (2 points) Let $S$ be an arbitrary multiplicative monoid. If $x \in S$ is prime, prove that $x \in S$ is irreducible.

## 3 Characterizing Primes and Irreducibles in H

**Problem 6a:** (9 points) Let $x \in \mathbf{H}$, $x \neq 1$. Prove that $x$ is irreducible in $\mathbf{H}$ if and only if either

1. $x = p$ where $p$ is prime in $\mathbb{Z}$ and $p \equiv 1 \pmod 4$
2. $x = p_1 p_2$ where $p_1$ and $p_2$ are prime in $\mathbb{Z}$ and we have that both $p_1 \equiv 3 \pmod 4$ and $p_2 \equiv 3 \pmod 4$.

**Problem 6b:** (6 points) Use problem 6a to show that the irreducibles of type 1 are prime in $\mathbf{H}$ and the irreducibles of type 2 are not prime in $\mathbf{H}$.

This tells us that in $\mathbf{H}$, factorization occurs differently than in $\mathbb{Z}$.

## 4 Characterizing Primes and Irreducibles in M

We now turn our attention back to $\mathbf{M}$, and characterize some properties about its elements:

**Problem 7:** (4 points) Let $x \in \mathbb{N}$ and $x \neq 1$. Prove that $x \in \mathbf{M}$ if and only if $x \equiv 0 \pmod 2$ and $x \equiv 1 \pmod 3$.

**Problem 8:** (4 points) If $x \neq 1$ and $x \in \mathbf{M}$, where $x = 2^k w$, with $w$ odd and $k \geq 3$, prove that $x$ is not irreducible in $\mathbf{M}$.

We now characterize the irreducibles of $\mathbf{M}$.

**Problem 9a:** (14 points) If $x \neq 1$ and $x \in \mathbf{M}$, then $x$ is irreducible if and only if both

1. $x = 2r$ where $r$ is an odd number and $r \equiv 2 \pmod 3$
2. $x = 4s$ where $s = 1$ or $s$ is the product of odd primes (here we mean primes in $\mathbb{N}$), each of which is equivalent to 1 modulo 3.

**Problem 9b:** (3 points) Prove that if $x \in \mathbf{M}$ is irreducible, then it is not prime in $\mathbf{M}$.

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This allows us to resolve some of our questions about irreducibility and primality in multiplicative monoids. This is what we have shown:

- In every multiplicative monoid, $x$ prime implies $x$ irreducible.
- In $\mathbb{Z}$, an element is irreducible if and only if it is prime.
- In $\mathbf{H}$, some irreducibles are prime, and some irreducibles are not prime.
- In $\mathbf{M}$, there are no irreducibles which are also prime.

Special thanks to Dr. Scott Chapman for allowing us to adapt his article for this tournament.