# Johns Hopkins Math Tournament 2018

# Proof Round: Fuzzy Knights and Knaves

*February 17, 2018*

| Section | Total Points | Score |
|---|---|---|
| 1 | 25 | |
| 2 | 20 | |
| 3 | 55 | |

## Instructions

- The exam is worth 100 points; each part's point value is given in brackets next to the part.
- To receive full credit, the presentation must be legible, orderly, clear, and concise.
- If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "explain," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says justify or prove, then you must prove your answer rigorously.
- Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa.
- Pages submitted for credit should be **numbered in consecutive order at the top of each page** in what your team considers to be proper sequential order.
- **Please write on only one side of the answer papers.**
- Put the **team number** (NOT the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

# 1 Introduction

In a distant planet, there is the city of knights and knaves, where all inhabitants are of two types: knight or knave. Knights only make true statements, and knaves only make false statements, but in all other ways, knights and knaves are identical. So, for an outsider, the only way to determine whether a native is a knight or a knave is to logically analyze the statements they make.

The following problem is an example of how to analyze the statements made by natives.
**Problem 0:** (0 points)

*Andrew: Ben and I are both knaves.*

Determine the types of Andrew and Ben.

**Proof.** Andrew is a knave and Ben is a knight. Andrew cannot possibly be a knight because then he would be lying about his own identity, so Andrew is a knave. But since Andrew is a knave, this statement must be false, so Ben must be a knight.

Now that you have a feeling for how things work, try the following problems. Note: for this section, we refer to "type" as the categories knight and knave.
**Problem 1:** (3 points)

*Andrew: We are the same type (both knights or both knaves).*

*Ben: We are of different types.*

What can you conclude about Andrew and Ben? Explain.

**Problem 2:** (4 points)

*Connie: There are exactly two knights among us.*

*David: If Connie's statement is true, then I am one of the knights.*

*Elysia: But it isn't true because Connie is a knave.*

What can you conclude about Connie, David, and Elysia? Explain.

**Problem 3:** (5 points)

*Felix: Gary and Hugh are knights.*

*Gary: Either Hugh is a knight or Felix is a knave.*

*Hugh: Gary is not a knave.*

What can you conclude about Felix, Gary, and Hugh? Explain.

**Problem 4:** (5 points)

*Ian: It's false that Jacob is a knave.*

*Jacob: Ian and I are different.*

What can you conclude about Ian and Jacob? Explain.

**Problem 5:** (5 points)

*Kiki: I am a knight or Lawrence is a knave.*
*Lawrence: I know that Kiki is a knight and that Mindy is a knave.*
*Mindy: Lawrence and I are different.*
What can you conclude about Kiki, Lawrence, and Mindy? Explain.

**Problem 6:** (3 points)

*Nick: I am a knave.*
Does this statement make sense? Explain.

## 2 Boolean Algebra

In the next section we'll be using a bit of boolean algebra, and so here we will introduce the symbols and notation used.

**Definition 1** *A* ***proposition*** $P$ *is a mathematical statement.*

**Example 1** *The statements "3 is prime" and "24 is odd" are propositions. The first one is true and the second one is false.*

**Definition 2** *Given a proposition $P$, the negation of a proposition, denoted $\neg P$, is true when $P$ is false and false when $P$ is true.*

**Definition 3** *The logical AND, denoted $\wedge$, is an operator that takes two propositions as input. Given propositions $A$ and $B$, the logical AND $A \wedge B$ is true if and only if $A$ and $B$ are true, and false otherwise.*

**Definition 4** *The logical OR, denoted $\vee$, is an operator that takes two propositions as input. Given propositions $A$ and $B$, the logical OR $A \vee B$ is true if one of $A$ or $B$ are true, and false otherwise.*

**Problem 7:** (6 points) Let $A, B$ be propositions. Fill in the following table, where values for $A, B$ are provided.

| $A$ | $B$ | $\neg A$ | $A \wedge B$ | $A \vee B$ |
|---|---|---|---|---|
| True | True | | | |
| True | False | | | |
| False | True | | | |
| False | False | | | |

The three operations $\wedge, \vee, \neg$ are

- commutative ($A \vee B = B \vee A$ etc.)
- associative ($A \vee (B \vee C) = (A \vee B) \vee C$)
- distributive, where $A \wedge (B \vee C) = (A \vee B) \wedge (A \vee C)$ and $A \vee (B \wedge C) = (A \wedge B) \vee (A \wedge C)$.

There are two more logical symbols we'll use: $\rightarrow$ and $\leftrightarrow$. We model statements of the form "If A, then B" by $A \rightarrow B$ and statements of the form "A if and only if B" by $A \leftrightarrow B$.

**Definition 5** *The conditional operator $\rightarrow$ and biconditional operator $\leftrightarrow$ are defined using the following truth table.*

| $A$ | $B$ | $A \rightarrow B$ | $A \leftrightarrow B$ |
|---|---|---|---|
| *True* | *True* | *True* | *True* |
| *True* | *False* | *False* | *False* |
| *False* | *True* | *True* | *False* |
| *False* | *False* | *True* | *True* |

**Problem 8a:** (3 points) Using only the operations $\wedge, \vee, \neg$, derive an expression equivalent to $A \rightarrow B$. Justify your answer.
**Problem 8b:** (3 points) Using only the operations $\wedge, \vee, \neg$, derive an expression equivalent to $A \leftrightarrow B$. Justify your answer.
**Problem 9a:** (4 points) Prove $A \rightarrow B$ and $(\neg B) \rightarrow (\neg A)$ are logically equivalent.
**Problem 9b:** (4 points) Prove $A \leftrightarrow B$ and $(A \rightarrow B) \wedge ((\neg A) \rightarrow (\neg B))$ logically equivalent.

## 3 Fuzzy Islands

Traditional knights and knaves, as explored above, is based on classical principles of logic, where there are two truth values (true/false) and they are mutually exclusive and exhaustive. But this is somewhat problematic because having a strict separation between true and false can't deal with vagueness.

On a "fuzzy" island, knighthood and knavehood are no longer permanent and islanders transition between the two states throughout their life. They might be knights for some time, and then transition to being a knave. During transition, they're partly knights and partly knaves, but once the transition is done, they are knaves. Then this process could reverse and the islander becomes a knight again.

Since people can now be partly knave and partly knight, their statements can't be completely true or false anymore – the "truthiness" or truth value of a statement is now a real number between 0 and 1. A truth value of 0 means a statement is false, and a truth value of 1 means a statement is true.

**Example 2** *If the statement "Jacob is a knight" is assigned a truth value of 0.9, then Jacob is in the transition period and is closer to being a knight than a knave.*

How can we model the truth value of compound statements such as "Andrew and Paul are knaves if Gerry is a knight or Kevin is a knave" on a fuzzy island?

**Definition 6** *Let $P$ be an arbitrary proposition. Then, $v(P)$ is the **truth value** of $P$.*

**Definition 7** *Let $P, Q$ be arbitrary propositions. Then we define the interaction of the truth value $v$ and logical operators $\neg, \wedge, \vee,$ as follows:*

$$\begin{aligned} v(\neg P) &= 1 - v(P) \\ v(P \wedge Q) &= \min(v(P), v(Q)) \\ v(P \vee Q) &= \max(v(P), v(Q)) \\ v(P \rightarrow Q) &= \min(1, 1 - (v(P) - v(Q))) \end{aligned}$$

For the conditional, the idea is that if $P$ is less true than $Q$, the conditional $P \rightarrow Q$ has truth value 1. If $P$ is more true than $Q$, the conditional has some flaw in truth, so we penalize the conditional by the difference in truth value between $P$ and $Q$.

The existence of the transition period means we need to redefine the social hierarchy on the fuzzy island. A recent knight who is less than halfway through his transition is considered a "quasiknight," and makes statements with high truth values (greater than 0.5). In the latter half of his transition, he becomes a "quasiknave" and makes statements with low truth values (less than 0.5). The opposite is true for a knave entering transition. Less than halfway through transition, he is a "quasiknave" and when more than halfway through transition he is a "quasiknight."
Specifically if $P$ is a proposition spoken by an islander A, then

$$\begin{cases} v(P) = 1 & \text{if } A \text{ is a knight (or fully knight).} \\ 0.5 < v(P) < 1 & \text{if } A \text{ is a quasiknight.} \\ 0 < v(P) < 0.5 & \text{if } A \text{ is a quasiknave.} \\ v(P) = 0 & \text{if } A \text{ is a knave (or fully knave).} \end{cases}$$

Note that no islander can ever make a statement with truth value 0.5.
For these problems, we will refer to "type" as the categories fully knight, quasiknight, quasiknave, and fully knave.

**Problem 10:** (7 points)

*Olivia: Paul is a quasiknight.*

*Paul: Olivia is a quasiknave.*

What can you conclude about Olivia and Paul? Explain.

**Problem 11:** (7 points)

*Qing: Richard is a knight.*

*Richard: Holy moly! Qing's statement has truth value 0.8.*

What can you conclude about Qing and Richard? Explain.

**Problem 12:** (7 points)

Provide (with proof) a sentence that no one on the island can say.

**Problem 13:** (7 points)

Suppose I tell you my friend Stanford, an islander, said, "I am a knave or a quasiknight." What would you conclude from this statement? Prove your conclusion.

For the following problem, we refer to "rank" of an islander. The closer someone is to being fully knight, the higher their rank. Someone fully knight is higher rank than a quasiknight, which is higher rank than a quasiknave, which is higher rank than a fully knave islander.

**Problem 14:** (9 points)

*Tommy: Valeera is a knave.*

*Uther: By the holy light! Valeera is a knave and Tommy is a knight.*

*Valeera: Watch your back. Tommy is lower rank than Uther.*

What can you conclude about Tommy, Uther, and Valeera? Explain.

**Problem 15:** (9 points)

*You meet Will, who says, "If I am a knight, then I am a knave."*

What can you conclude about Will? Explain.

**Problem 16:** (9 points)

*Xandra: Yonatan is a knight. Also, Zenny is a knave.*

*Yonatan: If I am a knave, then Zenny is a quasiknight.*

*Zenny: If I am a quasiknave, then Xandra and Yonatan are the same type.*

What can you conclude about Xandra, Yonatan, and Zenny? Explain.

Special thanks to Dr. Jason Rosenhouse for allowing us to adapt his article for this tournament.