# Johns Hopkins Math Tournament 2018

# Individual Round: General I

*February 17, 2018*

## Instructions

- **DO NOT TURN OVER THIS PAPER UNTIL TOLD TO DO SO.**
- This test contains 10 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Only answers written on the appropriate area on the answer sheet will be considered for grading.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, please submit your protest in writing to the JHMT Headquarters in **Bloomberg 276**.
- Good luck!

1. Finn, Rey, and Poe are each wearing helmets. The helmets each have two lightsabers drawn on them, and the lightsabers can either be both blue, both red, or one blue and one red. However there are no more than *four* of each color lightsaber in total. Luke goes around the circle asking each of them whether they know what color lightsabers are on their helmets. They answer as follows:

   Finn: "No."

   Rey: "No."

   Poe: "No."

   Finn: "Yes."

   How many red lightsabers are on Rey's helmet?

2. Find the sum of all possible integers $n$ such that $n^4 + 6n^3 + 11n^2 + 3n + 1$ is a perfect square.

3. The sum of the squares of two numbers is 2018 and the product of the two numbers is 49. One of the numbers is of the form $a + b\sqrt{c}$, where $a, b, c$ are positive integers. Find $a + b + c$.

4. Given that $xy + 4x + 2y = 17$, where $x$ and $y$ are both positive integers, find the sum of $x$ and $y$.

5. Assume $\alpha$, $\beta$, $\gamma$ satisfy $0 < \alpha < \beta < \gamma < 2\pi$, and

$$\cos(x + \alpha) + \cos(x + \beta) + \cos(x + \gamma) = 0$$

   for any real value $x$. Then we have that

$$\gamma - \alpha = \frac{a}{b}\pi$$

   where $a$ and $b$ are in lowest terms. What is $a + b$?

6. Let $m$ be the area and let $n$ be the perimeter of a regular octagon. The ratio $\frac{m^2}{n}$ can be expressed as $p\tan(q\pi)$ where $p$ is a positive integer. Find $pq$.

7. An equilateral triangle $ABC$ is in between two parallel lines $x, y$ that pass through points $A$ and $B$ respectively. Given that $C$ is twice as far from $y$ as $x$, the acute angle that $CA$ makes with $x$ is $\theta$. Then $(\tan\theta)^2$ is of the form $\frac{p}{q}$ where $p, q$ are relatively prime positive integers. Find $p + q$.

8. In how many ways can three distinct increasing integers be chosen from the range $[1, 100]$ such that they form an arithmetic progression?

9. The probability that a randomly chosen positive divisor of $2018^{2018}$ is a multiple of $2018^{18}$ is $(\frac{p}{q})^2$. Find $p + q$.

10. The square of an eight-digit integer is subtracted from the square of another eight-digit integer. The difference is sometimes an integer with eight identical even digits. What is the sum of the possible identical even digits in the difference?

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1. There are 70 Harry Potter fans in a room. 47 of them like Harry, 32 of them like Hermione, 24 of them like Ron, 8 of them like all three characters, and 9 of like none of the three characters. If 20 fans like both Harry and Hermione, and 17 fans like both Harry and Ron, how many fans like only Hermione?

2. If $S$ is a $100 \times 100$ square that is split into unit squares, then the diagonal of $S$ will pass through 100 unit squares. If $T$ is a $2018 \times 2022$ square that is split into unit squares, then how many unit squares will the diagonal of $T$ pass through?

3. Triangle $ABC$ satisfies the following two conditions:

$$2 \sin A + 5 \cos B = 7$$
$$5 \sin B + 2 \cos A = 2\sqrt{5}$$

   What is the measure of angle $C$, in degrees?

4. Two pairs of vertices of a regular pentacontagon (50-sided polygon) are randomly chosen. If each pair is connected by a line segment, the probability that these two lines intersect is $\frac{p}{q}$, where $p, q$ are relatively prime positive integers. Find $p + q$.

5. Two parallel chords of a circle have lengths 10 and 14 and are 6 units apart. The chord that is midway between these two chords and parallel to both has length $a$. What is $a^2$?

6. 83 math students labeled $1 - 83$ sit around in a circle. Starting from the student labeled 1, every 3rd student is eliminated (thus the student labeled 3 will be the first to leave). What is the label of the last student who remains?

7. Let $x$ and $y$ be positive numbers such that $\log_y(x) + \log_x(y) = \frac{10}{3}$ and $xy = 144$. Find $\frac{(x+y)^2}{4}$.

8. Two integers $a$ and $b$ are chosen uniformly at random from the first 100 positive integers. The probability that $3^a + 7^b \equiv 8 \pmod{10}$ is $\frac{a}{b}$ where $a, b$ are relatively prime positive integers. What is $a + b$?

9. 7 points are equally spaced around a circle. 4 chords that join some of these points are randomly created. The probability that these 4 chords form a quadrilateral is $ab$ where $a, b$ are relatively prime positive integers. What is $a + b$?

10. Let $R$ and $T$ be positive numbers that satisfy the equation

$$\log_9(R) = \log_{12}(T) = \log_{16}(R + T)$$

   The fraction $\frac{T}{R}$ can be expressed as $\frac{a+\sqrt{b}}{c}$ where $a, b, c$ are integers and $b$ is square-free. What is $a+b+c$?