Johns Hopkins Math Tournament 2018

Individual Round: Combinatorics

February 17, 2018

Instructions

- **DO NOT TURN OVER THIS PAPER UNTIL TOLD TO DO SO.**
- This test contains 10 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Only answers written on the appropriate area on the answer sheet will be considered for grading.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, please submit your protest in writing to the JHMT Headquarters in **Bloomberg 276**.
- Good luck!

1. There are m magic cards and t trap cards in a Pot of Greed. Pot of Greed lets you draw two cards (without replacement). When you draw two cards from Pot of Greed, the probability that both cards are magic cards is $\frac{1}{2}$. If t is even, what is the minimum value of m such that this is possible?

2. Téa randomly chooses a positive divisor of 2018^{2018}. The probability that her chosen number is a multiple of 2018^{18} is $(\frac{p}{q})^2$. Find $p+q$.

3. Kaiba doesn't think Yugi is the king of *all* games, so he challenges Yugi to a game where they make moves by uniformly choosing numbers at random along the interval [0,1]. Kaiba lets Yugi go first. Yugi and Kaiba keep a running sum of the numbers they have chosen, and if their running sum ever has fractional part less than or equal to 0.1, the person who made the final move wins. The probability that Kaiba will win the game is $\frac{p}{q}$ where p and q are relatively prime positive integers. Find $p+q$.

4. In a Shadow Game, Joey is required to choose 2018 positive integers $s_1, s_2, \ldots, s_{2018}$ such that the quotients of consecutive s_i, that is, numbers of the form $\frac{s_1}{s_2}, \frac{s_2}{s_3}, \ldots, \frac{s_{2017}}{s_{2018}}$, are all distinct. If Joey uses more distinct integers than needed, Bakura will take Joey's soul. What is the fewest number of distinct integers needed in the set $\{s_1, s_2, \ldots, s_{2018}\}$?

5. Kaiba still isn't convinced that Yugi is the king of games, so they play another game. The integers 1 to n are written in consecutive order on a blackboard. Starting with Kaiba, the two players alternate erasing any pair of adjacent numbers and replace it with either their sum or their product (in the same location). If the final number on the board is odd, Kaiba wins, but otherwise, Yugi wins. Find the 2018th smallest positive integer value for n such that Kaiba can guarantee victory.

6. Time Wizard chooses 18 random points uniformly on the edge his clock (which is a circle of radius 1), and connects them clockwise to form an 18-sided polygon. The probability that this polygon contains the center point of the circle is $\frac{p}{q}$, where p, q are relatively prime positive integers. Find p.

7. Mai Valentine has arranged 9 pieces of chocolate in a 3×3 grid. Each day, she eats one piece of chocolate, but she can only eat pieces that are not surrounded on all four sides (up, down, left, right). Assume that each day, each available piece of chocolate has an equal chance of being eaten. The expected number of days it takes to eat the middle piece of chocolate is N. Find $10N$.

8. Yugi's grandpa, Solomon, runs a card shop that only takes cash, and all his cards cost \$1. One day at his shop, he starts with no change in his cash register, 10 customers arrive at different times with either a \$1 bill or a \$2 bill (with equal chance). The probability he can give exact change to all 10 customers is $\frac{p}{q}$ where p, q are relatively prime positive integers. Find $p+q$.

9. Kaiba challenges Yugi to one last game – four-dimensional tic-tac-toe. How many ways are there to win a game of four-dimensional tic-tac-toe? (This is equivalent to: How many lines pass through three lattice points (a_1, a_2, a_3, a_4) in $\mathbb{R}^4$ where each a_i is in $\{1, 2, 3\}$?)

10. Duke Devlin randomly labels the vertices of a regular 9-gon with the numbers 1 through 9. Let S be the sum of the absolute values of the differences of the numbers on adjacent vertices. Let the probability that S attains its minimum value be $\frac{p}{q}$ where p, q are relatively prime positive integers. Find $p+q$.