Johns Hopkins Math Tournament 2018

Individual Round: Calculus

February 17, 2018

Instructions

- **DO NOT TURN OVER THIS PAPER UNTIL TOLD TO DO SO.**
- This test contains 10 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Only answers written on the appropriate area on the answer sheet will be considered for grading.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, please submit your protest in writing to the JHMT Headquarters in **Bloomberg 276**.
- Good luck!

1. Compute
$$\lim_{n\to\infty}\int_{-1}^{1}\sqrt[n]{1-x^n}\,dx.$$

2. Let two unit spheres be centered around $(0,0,0)$ and $(0,0,1)$. Let the volume of intersection be $\frac{p}{q}\cdot\pi$. Find $p+q$.

3. If
$$\int_{\pi}^{2\pi}\sin^4(x)\,dx=\frac{a}{b}\pi$$
where a and b are relatively prime positive integers, find $a\cdot b$.

4. A circle centered at $(4,0)$ is tangent to the curve $y=x^3+1$ at the point (x_0,y_0). The sum of all possible values of x is $\frac{p}{q}$ where p,q are relatively prime positive integers. Find $p+q$.

5. Given
$$\int_0^{\infty}\frac{dx}{1+x^3}=\frac{a\pi\sqrt{3}}{b},$$
where a and b are relatively prime positive integers, find $a+b$.

6. Given
$$\int_0^8\frac{1}{\sqrt{1+\sqrt{1+\sqrt{1+x}}}}\,dx=\frac{a+b\sqrt{2}}{c}$$
where the RHS in least terms (i.e. $\gcd(a,b,c)=1$), find $a+b+c$.

7. Suppose
$$\frac{1+\frac{1}{2^{11}}+\frac{1}{3^{11}}+\frac{1}{4^{11}}+\frac{1}{5^{11}}+\ldots}{1-\frac{1}{2^{11}}+\frac{1}{3^{11}}-\frac{1}{4^{11}}+\frac{1}{5^{11}}-\ldots}=\frac{a}{b}$$
where a and b are relatively prime positive integers. Find $a+b$.

8. Evaluate
$$\lim_{x\to\infty}\left[(x^6+x^5)^{\frac{1}{6}}-(x^6-x^5)^{\frac{1}{6}}\right].$$

9. Let C be the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}$ where $x,y\geq 0$. Find the length of the longest line segment that lies in the first quadrant and is tangent to C.

10. Let $\frac{p}{q}$ be the area of the region bounded by $y=x^{2018}$ and $y=x^{1/2018}$ and lying above the x-axis, where p,q are relatively prime positive integers. Find $p+q$.