Johns Hopkins Math Tournament 2018

Individual Round: Algebra

February 17, 2018

Instructions

- **DO NOT TURN OVER THIS PAPER UNTIL TOLD TO DO SO.**
- This test contains 10 questions to be solved individually in 60 minutes.
- All answers will be integers.
- Only answers written on the appropriate area on the answer sheet will be considered for grading.
- Problems are weighted relative to their difficulty, determined by the number of students who solve each problem.
- No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.
- If you believe the test contains an error, please submit your protest in writing to the JHMT Headquarters in **Bloomberg 276**.
- Good luck!

1. Find the sum of all possible integers n such that $n^4 + 6n^3 + 11n^2 + 3n + 1$ is a perfect square.
2. Let $z > 0$ such that $z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = 11$. Compute $z^8 + z^4 + 1 + \frac{1}{z^4} + \frac{1}{z^8}$.
3. Compute the number of factors of 16005600 that have an even amount of prime factors.
4. Let N be a 3-digit integer such that the difference between any two positive factors of N is divisible by 3. Let $d(N)$ be the number of positive divisors of N. Compute the maximum possible value of $N \cdot d(N)$.
5. Let the sequence $x_1, x_2, \ldots$ be such that $x_1 = 7$, and
$$x_n = x_{n-1} + \left(\frac{2x_{n-1} - (n+2)}{n}\right) + 1$$
for $n \geq 2$. Find x_{40}.
6. Let S be a set of complex numbers such that if $u, v \in S$, then $uv \in S$ and $u^2 + v^2 \in S$. If there are k elements of S with absolute value at most 1, and $k < \infty$, what is the largest value of k.
7. Let $a_1, a_2, \ldots a_{18}$ be real numbers such that $\sum_{i=1}^{17} (a_{i+1} - a_i)^2 = 1$. Find the maximal value of $(a_{10} + a_{11} + \cdots + a_{18}) - (a_1 + a_2 + \cdots + a_9)$.
8. Suppose a set S satisfies the following conditions:
   (1) every element in S is a positive integer and not greater than 100;
   (2) for any two different elements $a, b \in S$, there is an element $c \in S$ such that $\gcd(a, c) = \gcd(b, c) = 1$;
   (3) for any two different elements $a, b \in S$, there is an element d, not equal to a or b, such that $\gcd(a, d) > 1$ and $\gcd(b, d) > 1$.

   Find the maximum number of elements in S.
9. Given that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{6}{5} \cdot Q$ for some Q, then
$$\sum_{\substack{k=1 \\ k \text{ is not divisible by } 5}}^{\infty} \frac{1}{k^3} = \frac{m}{n} \cdot Q$$
where m and n are relatively prime positive integers. Find $m - n$.
10. Suppose that α and β are different real roots of the equation $4x^2 - 4tx - 1 = 0$, for a real-valued t. Define two functions $f(x) = \frac{2x-t}{x^2+1}$, where $f(x)$ has domain $[\alpha, \beta]$, and $g(t) = \max f(x) - \min f(x)$. Then, $g(5)^2 = \frac{a}{b}$, where a, b relatively prime positive integers. Find $a + b$.