Global Stability for Charged Scalar Fields in Spacetimes close to Minkowski

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1. Introduction
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3. The Proof
As motivation for this work, we consider the Einstein field equations with zero cosmological constant,

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8 \pi Q_{\mu\nu}, \]  

(1)

where \( R_{\mu\nu} \) is the Ricci curvature associated with the metric \( g \), \( R \) is the scalar curvature, and \( Q \) is the energy-momentum tensor associated with some set of field equations.
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where $R_{\mu\nu}$ is the Ricci curvature associated with the metric $g$, $R$ is the scalar curvature, and $Q$ is the energy-momentum tensor associated with some set of field equations. The simplest such model is the Einstein vacuum equations ($Q_{\mu\nu} = 0$), which can be written as

$$R_{\mu\nu} = 0.$$
Properties of the Einstein Vacuum Equations

This system is of interest from a mathematical standpoint for several reasons:

- It admits nontrivial solutions. Two sets of famous exact solutions are the Schwarzschild and Kerr solutions, which model black hole spacetimes.
- This system linearizes to the wave equation around the Minkowski spacetime. That is, for a choice of coordinates we can write the system as
  \[ g_{\alpha\beta} \partial_\alpha \partial_\beta (g_{\mu\nu}) = P_{\mu\nu} (g) (\partial g, \partial g), \]
  for a quadratic form \( P_{\mu\nu} (g) \).
- Finally, the background spacetime for solutions of Einstein's field equations can often be modelled by solutions of the vacuum equations. This means we can in a sense treat the equations modelling the field and spacetime separately.
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Solutions to Einstein’s vacuum equations with small initial data can be modelled by the metric

\[ g_{\alpha\beta} = m_{\alpha\beta} + \frac{M\chi}{1 + r}\delta_{\alpha\beta} + h_{\alpha\beta}, \]

where \( m \) is the Minkowski metric, \( \frac{M\chi}{1 + r}\delta_{\alpha\beta} \) is spherically symmetric (where \( \delta \) is the Kronecker delta and \( M \) is a small parameter corresponding to the mass), and \( h \) is a small perturbation satisfying certain decay bounds. We have by the positive mass theorem that \( M > 0 \).
The Background Spacetime

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The first two terms imply that the metric behaves like the Schwarzschild metric in the far exterior. The third term is naturally of the most interest when establishing global stability results for the Einstein Vacuum Equations.
We choose as our set of field equations the Maxwell-Klein-Gordon system,

\[ \Box_g \phi = D^\alpha D_\alpha \phi = 0, \quad (2a) \]
\[ \nabla^\beta F_{\alpha\beta} = \mathfrak{I}(\phi D_\alpha \phi), \quad (2b) \]

where, for a one-form \( A \), we define

\[ F = dA, \quad D_\mu = \nabla_\mu + iA_\mu. \]

Additionally, we define the current vector

\[ J_\alpha = \mathfrak{S}(\phi D_\alpha \phi). \]
The Main Theorem

We are ready to state our main theorem:

**Theorem**

Given a background metric $g$ satisfying bounds consistent with small-data solutions of the Einstein Vacuum Equations in harmonic gauge, the system (2) is globally well-posed for small initial data. Additionally, given some initial energy on $k$ derivatives, $\mathcal{E}_k[F, \phi]$, with $k \geq 11$, the energy-momentum tensor,

$$ Q[F, \phi]_{\alpha\beta} = \Re \left( D_{\alpha} \phi D_{\beta} \phi - \frac{1}{2} g_{\alpha\beta} D_{\gamma} \phi D_{\gamma} \phi \right) + F_{\alpha\gamma} F_{\beta}^{\gamma} - \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta}, \quad (3) $$

satisfies the following estimates:

$$ \| Q[F, \phi](t, \cdot) \|_{L^1} \lesssim \mathcal{E}_k[F, \phi] \quad (4) $$

$$ \| Q[F, \phi](t, \cdot) \|_{L^2} \lesssim \mathcal{E}_k[F, \phi](1 + t)^{-1} \quad (5) $$
The estimate on $\| Q[F, \phi](t, \cdot) \|_{L^2}$ follows directly from decay estimates (and is sharp in time decay in only one component of $Q$); however, this is in a sense the most important estimate for closing the argument for the full Einstein-Maxwell-Klein-Gordon system. In particular, in an energy estimate we can generally use this to establish slowly growing energy for the metric, which is consistent with results in (Lindblad and Rodnianski 2010). Estimates of this sort have been of great use in establishing global stability for the Einstein-Vlasov system in (Lindblad and Taylor 2017)
Comparison to Previous Work

- Global Stability and Decay for the MKG system
  - (Eardley and Moncrief 1982) and (Klainerman and Machedon 1994): Global existence for finite initial data in Coulomb gauge.
  - (Lindblad and Sterbenz 2006): Nice decay estimates in a gauge-free setting for small initial data.
  - (Bieri, Miao, and Shahshahani 2017): Provided a simpler proof of results found by Lindblad and Sterbenz.
  - (Yang 2015): Removed the smallness assumption on the Maxwell field.

The Einstein Field Equations

- (Choquet-Bruhat 1952): Local stability of the Minkowski spacetime
- (Christodoulou and Klainerman 1990): Global stability of the Minkowski spacetime
- (Lindblad and Rodnianski 2010): Global stability of the Minkowski in harmonic coordinates
- (Zipser 2000): Global Stability for the Einstein-Maxwell system
- (Loizelet 2008): Global Stability for Einstein Maxwell in Lorenz gauge harmonic coordinates
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The Lorentz Vector Fields

It is well-known that for the vector fields

\[ \mathbb{L} = \{ \partial_\alpha, \Omega_{ij} = x^i \partial_j - x^j \partial_i, \Omega_{0i} = t \partial_i + x^i \partial_t, S = t \partial_t + r \partial_r \}, \]

if \( \Box u = f, \Box (Zu) = Zf + c_Z f \), where \( c_Z = 2 \) if \( Z = S \) and \( c_Z = 0 \) for all other fields in \( \mathbb{L} \).

This follows from the calculation

\[ [Z, \Box] \phi = - (\mathcal{L}_Z m)^{\alpha\beta} \partial_\alpha \partial_\beta \phi - (\Box Z^\gamma) \partial_\gamma \phi. \]

The first term on the right is a scalar multiple of \( \Box \phi \) which comes from the Killing and conformal Killing nature of the fields \( Z \), and the second term is 0, since all components of \( Z \) are at most first degree polynomials.
The Null Frame

One notable consequence of this is that certain derivatives of $\phi$ decay at different rates. In particular, we define

$$L = \partial_t + \partial_r, \quad \underline{L} = \partial_t - \partial_r,$$

as well as a set of piecewise defined derivatives $S_1, S_2$ which are tangent to the sphere. Then, roughly speaking, if $\phi$ decays like $t^{-1}$, we can generally expect the following estimates for derivatives of $\phi$:

$$|L\phi| + \sum_i |S_i\phi| \lesssim (1 + t)^{-2},$$

$$|\underline{L}\phi| \lesssim (1 + t)^{-1}(1 + |r - t|)^{-1}.$$

In particular, the latter has worse decay close to the light cone $t = r$. 
The Null Condition and Asymptotic Systems

As an application, we consider the equation

$$\Box u = - (\partial_t u)^2,$$

in $\mathbb{R}^{1+3}$, with the initial conditions

$$u(0, x) = \epsilon u_0(x), \quad \partial_t u(0, x) = \epsilon u_1(x), \quad u_0, u_1 \in C_0^\infty.$$

We can rewrite this as

$$\Box u = - \frac{1}{r} LL(ru) + \Delta_\omega u = - \frac{1}{4} \left( \frac{1}{r} L(ru) + \frac{1}{r} \overline{L}(ru) \right)^2.$$

where $\Delta_\omega$ denotes the spherical Laplacian. If we assume all derivatives except $\underline{L}$ are negligible, we get the equation

$$LL(ru) = \frac{1}{4r} |\underline{L}(ru)|^2.$$

Solutions of this equation, and the $1 + 3$-dimensional equation it models, can blow up in finite time, no matter how small the initial data!
For certain classes of quasilinear equations, we have global existence for small initial data. One famous example is the equation

\[ \square u = (\partial_t u)^2 - |\nabla_x u|^2, \]

which has global solutions for sufficiently small \( \epsilon \) (cf. (Klainerman 1986)). In our null decomposition, the right hand side is equal to

\[ Lu \cdot Lu - |\nabla u|^2, \]

i.e. the problematic term, \( |Lu|^2 \), is absent! The asymptotic system for this equation is

\[ LL(\rho u) = 0, \]

which of course has global existence for all time. This in general holds for systems where the right hand side is composed of null forms

\[ Q_0[\phi, \psi] = m^{\alpha \beta} \partial_\alpha \phi \partial_\beta \psi, \quad Q_{\alpha \beta}[\phi, \psi] = \partial_\alpha \phi \partial_\beta \psi - \partial_\alpha \psi \partial_\beta \phi. \]
For all quadratic nonlinearities satisfying the null condition, we have the asymptotic system

\[ LL(ru) = 0, \]

which leads to global existence. However, this is not the only case when the asymptotic system doesn’t blow up. Consider the system

\[ \Box \psi_1 = 0, \tag{6a} \]
\[ \Box \psi_2 = (\partial_t \psi_1)^2. \tag{6b} \]

This has the asymptotic system

\[ LL(r\psi_1) = 0, \]
\[ LL(r\psi_2) = \frac{1}{4r} |L(r\psi_1)|^2. \]
This has global solutions, with solutions growing like

\[(r\psi_1) \approx 1, \quad (r\psi_2) \approx \ln(r).\]

It is known that solutions to this reduction also model the asymptotic behavior of the full system (6). It was shown in the celebrated work (Lindblad and Rodnianski 2010), that the accuracy of this model extends to solutions to Einstein’s Equations.
The Modified Frame

Due to the $\frac{M\chi}{1+r}\delta_{\alpha\beta}$ term in the metric $g$, we cannot reasonably expect $g_{LL}$ to decay any better than $\epsilon t^{-1}$ along the light cone $t = r$. This error term leads to slowly increasing energy in the standard energy estimate (cf. (Lindblad and Rodnianski 2010)), and growing energy in the conformal Morawetz estimate. We account for this in two steps: First, we use the modified radial coordinate $r^* = r + M\chi \ln(1 + r)$, $t^* = t$ and null frame

$$L^* = \partial_t^* + \partial_{r^*}, \quad L^* = \partial_t^* - \partial_{r^*}, \quad S^*_i = \frac{r}{r^*} S_i,$$

where $S_i$ are piecewise defined orthonormal vectors tangent to the sphere. The use of $r^*$ here comes from similar analysis on the Schwarzschild metric in the exterior, and will be of use in bounding certain error terms stemming from the massive part of the metric.
The Modified Frame

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$$L^* = \partial_t^* + \partial_{r^*}, \quad L_0^* = \partial_t^* - \partial_{r^*}, \quad S_i^* = \frac{r}{r^*}S_i,$$

where $S_i$ are piecewise defined orthonormal vectors tangent to the sphere. The use of $r^*$ here comes from similar analysis on the Schwarzschild metric in the exterior, and will be of use in bounding certain error terms stemming from the massive part of the metric. Second, we carry out our energy estimates with respect to the fractional acceleration field,

$$\overline{K_0^s} = \frac{1}{2}(1 + |t + r^*|^{2s})L^* + \frac{1}{2}(1 + |t - r^*|^{2s})L_0^*. $$
In addition to the modified radial coordinate and frame, we also define \( x^*i = r^*\omega_i \) and consequently the Lorentz fields

\[
\mathbb{L}^* = \{ \partial_{x^*\alpha}, \Omega^*_{ij} = x^*i \partial_{x^*j} - x^*j \partial_{x^*i}, \Omega^*_0 = t \partial_{x^*i} + x^*i \partial_{t^*}, S^* = t \partial_{t^*} + r^* \partial_{r^*} \}
\]

We generally use \( Z \) to refer to any of these fields. These are analogous to the Lorentz fields in Minkowski space (and in fact the rotation fields are unchanged). This modification of the fields is in particular necessary in making sure null components of Lie derivatives like \([Z, L^*]\) have the right decay along the light cone.
The Maxwell-Klein-Gordon Equations: The Weak Null Condition

We recall the potential $A_\mu$. We have some freedom in the choice of gauge; in particular, we can assume $A$ satisfies the Lorenz gauge condition,

$$\nabla \cdot A = 0.$$ 

In Minkowski space, this reduces the Maxwell-Klein-Gordon system to a system of semilinear wave equations,

$$\Box A_\mu = \Im(\phi D_\mu \phi),$$  \hspace{1cm} (7)  

$$\Box \phi = -2iA^\mu \partial_\mu \phi + A^\mu A_\mu \phi.$$  \hspace{1cm} (8)

Intuitively, every time a bad derivative or component of $A$ appears on the right of (8), it is paired with a good derivative or component. In (7), this holds everywhere except for the component $A_L$, for which we can similarly expect decay like $(1 + t)^{-1} \ln(1 + t)$ along the light cone.
Properties of the Metric

We recall the assumption that the metric is of the form

$$g_{\alpha\beta} = m_{\alpha\beta} + \frac{M\chi}{1+r} \delta_{\alpha\beta} + h_{\alpha\beta}.$$ 

Here $m$ is the Minkowski metric, $M \leq \epsilon$, and $h$ satisfies the $L^\infty$ estimates

$$|\mathcal{L}_Z^I h| \leq \epsilon \tau_+^{1+\iota},$$
$$|\mathcal{L}_Z^I h_{\mathcal{L}T}| \leq \epsilon \tau_+^{1-\gamma'+\iota} \tau_-, $$

for $\tau_+ = |t + r^*|$, $\tau_- = |t - r^*|$, $\mathcal{L} \in \{L^*\}$, $\mathcal{T} = \{L^*, S_1^*, S_2^*\}$. We take as well the $L^2$ estimates

$$\left\| \tau_+^{-1/2-\iota}(|\partial \mathcal{L}_Z^I h| + \tau_-^{-1}|\mathcal{L}_Z^I h|)w_1^{1/2}\right\|_{L^2(t,x)} \leq \epsilon,$$
$$\left\| \tau_-^{-1/2}\tau_+^{1-\iota}(|\partial \mathcal{L}_Z^I h|_{\mathcal{L}L} + |\overline{\partial} \mathcal{L}_Z^I h| + \tau_-^{-1}|(\mathcal{L}_Z^I h)_{\mathcal{L}L}|)w_1^{1/2}\right\|_{L^2(t,x)} \leq \epsilon,$$

for the simplified weight $w_1 = \langle (r^* - t)_+ \rangle$, and $Z \in \mathbb{L}^*$. 

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Properties of the Metric (Cont.)

The nicer estimates on $h_{LT}$ and $h_{LL}$ are a consequence of the harmonic coordinate condition. In particular, from the harmonic coordinate condition we have the rough identification

$$L(g_{LT}) \approx \bar{\partial} g$$

These in particular give nicer estimates for the Lie derivative $(L_{K_0}^* g)_{L^* L^*}$, which are necessary to control the certain components in the energy estimate.

We can say equivalently that $u^* = t - r^*$ is an approximate optical function, in the sense that we have nice decay for the quantity

$$g^{\alpha \beta} \partial_{\alpha} u^* \partial_{\beta} u^*.$$

This decay implies that we should look at the geometry as a small perturbation of Schwarzschild in the exterior, rather than of Minkowski, in order to get the best control over our field quantities.
In addition to the fractional acceleration field, our proof also heavily relies on the use of the following weights:

\[
  w = \begin{cases} 
    1 & r - t < 0 \\ 
    \langle r - t \rangle^{2\gamma} & r - t > 0 
  \end{cases}
\]

\[
  w' = \begin{cases} 
    \langle r - t \rangle^{-1-2\iota} & r - t < 0 \\ 
    \langle r - t \rangle^{2\gamma-1} & r - t > 0 
  \end{cases}
\]

\[
  w_\iota = \begin{cases} 
    \langle r - t \rangle^{2\iota} & r - t < 0 \\ 
    \langle r - t \rangle^{2\gamma} & r - t > 0 
  \end{cases}
\]

We in particular use a weight \( \tilde{w} \), which we do not define here, such that the following hold:

\[
  \tilde{w} \approx w, \quad L(\tilde{w}) \approx w', \quad L(\tilde{w}) \approx \left( \frac{\langle t - r^* \rangle}{\langle t + r^* \rangle} \right)^{1+2\iota} w'.
\]
Optical Weights and the Null Decomposition of $F$

We additionally define the optical weights

$$\tau_+ = \langle t + r^* \rangle = (1 + (t + r^*)^2)^{1/2},$$
$$\tau_- = \langle t - r^* \rangle = (1 + (t - r^*)^2)^{1/2},$$
$$\tau_0 = \tau_- / \tau_+.$$

Additionally, we define the null decomposition of a two-form $F$:

$$\alpha_i[F] = F_{L^*s_i^*}, \quad \alpha_i[F] = F_{L^*s_i^*},$$
$$\rho[F] = \frac{1}{2} F_{L^*L^*}, \quad \sigma[F] = F_{S_1S_2}.$$

We have the following rough identification:

$$\alpha_i[F] \sim \frac{D_{L^*}(r^*\phi)}{r^*}, \quad \alpha_i[F] \sim D_{L^*}\phi,$$
$$\rho[F] \sim D_{S_i}\phi, \quad \sigma[F] \sim D_{S_i}\phi,$$

which comes from the fact that
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The Energy Estimate: \( F \)

We define the energy-momentum tensor for an arbitrary 2-form \( G \) as follows:

\[
Q[G]_{\alpha\beta} = G_{\alpha\gamma} G^\gamma_{\beta} - \frac{1}{4} g_{\alpha\beta} G_{\gamma\delta} G^{\gamma\delta}.
\]

For a general two-form \( G \) with sufficient decay, we can define the time-slice energy on \( G \)

\[
E_0[G](T) = \sup_{0 \leq t \leq T} \int_{\Sigma_t} \left( \tau_+^{2s} (|\alpha|^2 + |\rho|^2 + |\sigma|^2) + \tau_-^{2s} |\alpha|^2 \right) w \, dx,
\]

the interior energy

\[
S_0[G](T) = \int_{[0,t] \times \mathbb{R}^3} \left( \tau_+^{2s} |\alpha|^2 + \tau_0^{1+2s} (\tau_+^{2s} (|\rho|^2 + |\sigma|^2) + \tau_-^{2s} |\alpha|^2) \right) w' \, dx \, dt,
\]

and the light-cone energy

\[
C_0[G](T) = \sup_{u^*} \int_{\{C(u^*)\} \cap \{t \in [0,T]\}} Q(K_0^s, \nabla u) w \, dV_C(u^*).
\]
The Energy Estimate: $F$ (Cont.)

We have the following preliminary energy estimate:

**Theorem**

Given a 2-form $F_{\alpha\beta}$ such that $\nabla^\beta F_{\alpha\beta} = J_\alpha$, and defining the norm

$$\| J \|_{L^2[w]} = \left\| \tau_+ \tau_0^{-1/2-\iota} \tau_-^{1/2} J_{L^*} w^{1/2} \right\|_2 + \left\| \tau_+ \tau_-^{1/2} |J_{S^*}| w^{1/2} \right\|_2 +$$

$$+ \left\| \tau_0^{-s-1/2-\iota} \tau_-^{s+1/2} |J_{L^*}| w^{1/2} \right\|_2,$$

we have the estimate

$$E_0[F](T) + S_0[F](T) + C_0[F](T) \lesssim E_0[F](0) + \| J[F] \|^2_{L^2[w]}.$$  (9)

We briefly sketch the proof.
Proof.

We follow a standard divergence estimate, applied to the momentum density

\[ P_\alpha[F] = -Q[F]_{\alpha\beta} K_0^{s}\beta w. \]

First, we take the divergence theorem over either time slabs of the form \([0, T] \times \mathbb{R}^3\), or time slabs exterior to some light cone \(t - r = C\). The boundary terms coming from these identities in particular contain the \(E_0\) and \(C_0\) terms respectively.

We now consider the interior. First, we have

\[ -(\nabla^\alpha w) K_0^{s}\beta Q[F]_{\alpha\beta} \approx -S_0[F]. \]
The Energy Estimate: \( F \) (Proof Cont.)

Proof.

Next, we can discard certain terms coming from the deformation tensor \((\overline{K}_0^s)_{\pi}\), as they have the right sign due to the choice of field. The terms coming from the interior energy therefore behave like

\[
\left\| F_{K_0^s\alpha} J^\alpha w \right\|_{L^1(t,x)} + \epsilon \int_0^T \frac{E[F](t)}{(1 + t)^{1+\nu}}.
\]

We use Hölder’s inequality to deal with the first term on the left, splitting it into a term which can be subtracted off of \( S_0[F](T) \) times a term corresponding to the current norm, and note that the second term can be bounded by an arbitrarily small constant times the energy.
One issue that immediately crops up is that the energy norm \( E_0[F](0) \) is in general not finite. In particular, we look at the elliptic consideration in the Lorenz gauge

\[
\Delta A_0 = \partial_j (\partial_t A_j) + J_0,
\]

which follows almost directly from the Lorenz gauge condition. In particular, it is not possible for \( A_0 \) to decay better than \( r^{-1} \), or certain components of \( F \) to decay better than \( r^{-2} \), unless \( J_0 \) integrates to 0. Our method for accounting for this follows from (Lindblad and Sterbenz 2006), and involves subtracting off a well-defined charge quantity,

\[
\overline{F}_{0i} = \omega^i \left( \frac{q}{4\pi} \frac{\chi(r^* - t - 2) \partial_r (r^*)}{r^{*2}} \right).
\]

and running subsequent analysis on the quantity \( \widetilde{F} = F - \overline{F} \).
The Energy Estimate: $\phi$

We can take an energy estimate on the quantity $\phi$ analogous to that for $F$. We take the energies

$$E_0[\phi](T) = \sup_{0 \leq t \leq T} \int_\Sigma_t \left( \tau_+^{2s} \left( \left| \frac{D_{L^*}(r^*\phi)}{r^*} \right|^2 + |\mathcal{D}\phi|^2 + \left| \frac{\phi}{r^*} \right|^2 \right) + \tau_-^{2s} |D_{L^*}\phi|^2 \right) w \, dx,$$

as well as

$$S_0[\phi](T) = \int_0^T \int_\Sigma_t \left( \tau_+^{2s} \left| \frac{D_{L^*}(r^*\phi)}{r^*} \right|^2 \right) w' \, dx \, dt +$$

$$+ \int_0^T \int_\Sigma_t \left( \tau_0^{1+2i} \left( \tau_+^{2s} \left( |\mathcal{D}\phi|^2 + \left| \frac{\phi}{r^*} \right|^2 \right) + \tau_-^{2s} (|D_{L^*}\phi|^2) \right) \right) w' \, dx \, dt$$

and

$$C_0[\phi](T) = \sup_{u^*} \int_{C(u^*)} Q(\nabla^\alpha_{m^*} u^*, \overline{K}_0^s) w \, dC(u^*).$$
We now have the following:

**Theorem**

*For a function \( \phi \) with sufficient decay, and sufficient \( L^\infty \) decay for the field \( F \), we have the estimate*

\[
E_0[\phi](T) + S_0[\phi](T) + C_0[\phi](T) \lesssim E_0[\phi](0) + \left\| T^s_+ T^{1/2}_- (\Box_g \phi) w^{1/2}_i \right\|_2^2. \tag{10}
\]

We note that the \( L^\infty \) decay for \( F \) is consistent with the bounds in the next section, and only depends on the quantity \( F \). In particular, we can establish this estimate on higher derivatives of \( \phi \) without requiring higher derivatives on \( F \).
The Energy: Combined

We define the combined energies

\[ \mathcal{E}_0[F](T) = q[F]^2 + \mathcal{E}_0[\tilde{F}](T) + S_0[\tilde{F}](T) + C_0[\tilde{F}](T), \]
\[ \mathcal{E}_0[\phi](T) = \mathcal{E}_0[\phi](T) + S_0[\phi](T) + C_0[\phi](T). \]

Additionally, for a collection of vector fields \( Z \in \mathbb{L}^* \), we define

\[ \mathcal{E}_k[F](T) = q[F]^2 + \sum_{|l| \leq k} \mathcal{E}_0[L_Z^l \tilde{F}](T), \]
\[ \mathcal{E}_k[\phi](T) = \sum_{|l| \leq k} \mathcal{E}_0[D_Z^l \phi](T), \]

and the total energy

\[ \mathcal{E}_k(T) = \mathcal{E}_k[F](T) + \mathcal{E}_k[\phi](T) \]
The Decay Estimates: \( F \)

We now establish decay for \( F \). There are two parts to these calculations. First, we take decay for the charge-free portion \( \widetilde{F} \) using the energy estimate and weighted Klainerman-Sobolev estimates. In particular we get, for \( |I| \leq k - 5 \),

\[
|\alpha[\mathcal{L}_Z^l \widetilde{F}]| \lesssim \mathcal{E}_k^{1/2} (T) \langle t + r \rangle^{-3/2 - s} w^{-1/2},
\]

\[
|\rho[\mathcal{L}_Z^l \widetilde{F}]| \lesssim \mathcal{E}_k^{1/2} (T) \langle t + r \rangle^{-1-s} \langle t - r \rangle^{-1/2} w^{-1/2},
\]

\[
|\sigma[\mathcal{L}_Z^l \widetilde{F}]| \lesssim \mathcal{E}_k^{1/2} (T) \langle t + r \rangle^{-1-s} \langle t - r \rangle^{-1/2} w^{-1/2},
\]

\[
|\alpha[\mathcal{L}_Z^l \widetilde{F}]| \lesssim \mathcal{E}_k^{1/2} (T) \langle t + r \rangle^{-1} \langle t - r \rangle^{-1/2 - s} w^{-1/2}.
\]

The estimates on \( \rho, \sigma, \alpha \) come from weighted Klainerman-Sobolev estimates over the time-slice energy, \( E_0 \). The estimate on \( \alpha \) comes from the conical energy \( C_0 \). Additionally, there is an \( L^2(t)L^\infty(x) \) estimate for \( \alpha \) which we do not state here, which comes from the interior energy \( S_0 \).
We additionally have the following estimates on $\bar{F}$, for all sets of vector fields $I$ (and the implicit constant in $\lesssim$ depending on $|I|$):

\[
|\alpha [\mathcal{L}_Z^I \bar{F}]| \lesssim |q| \cdot \langle t + r \rangle^{-3} \langle t - r \rangle H(r - t),
\]
\[
|\rho [\mathcal{L}_Z^I \bar{F}]| \lesssim |q| \cdot \langle t + r \rangle^{-2} H(r - t),
\]
\[
|\sigma [\mathcal{L}_Z^I \bar{F}]| \lesssim |q| \cdot \langle t + r \rangle^{-2} H(r - t),
\]
\[
|\alpha [\mathcal{L}_Z^I \bar{F}]| \lesssim |q| \cdot \langle t + r \rangle^{-2} H(r - t).
\]

Here, $H$ is the Heaviside function, meaning that $\bar{F}$ is supported outside the light cone. Here we recall that $q$ is the charge.
The Decay Estimates: $\phi$

We can establish analogous estimates for $\phi$. We have

$$|\phi| \lesssim \mathcal{E}_5^{1/2}(T) \langle t + r \rangle^{-1} \langle t - r \rangle^{1/2-s} w^{-1/2},$$

$$|D_{L^*} \phi| \lesssim \mathcal{E}_5^{1/2}(T) \langle t + r \rangle^{-1} \langle t - r \rangle^{-1/2-s} w^{-1/2},$$

$$|D_{S^*} \phi| \lesssim \mathcal{E}_5^{1/2}(T) \langle t + r \rangle^{-1-s} \langle t - r \rangle^{-1/2} w^{-1/2},$$

$$|D_{L^*} \phi| \lesssim \mathcal{E}_5^{1/2}(T) \langle t + r \rangle^{-1-s} \langle t - r \rangle^{-1/2} w^{-1/2},$$

$$\left| \chi \frac{D_{L^*}(r^* \phi)}{r^*} \right| \lesssim \mathcal{E}_5^{1/2}(T) \langle t + r \rangle^{-3/2-s} w^{-1/2}.$$

The proof for these is similar to analogous estimates for $F$. In particular, the first four follow from the Klainerman-Sobolev inequality applied to the time slice, with a Poincare-type estimate to handle the terms where no derivatives fall on $\phi$. The last estimate follows from a Klainerman-Sobolev type estimate using the conical energy $C_0$. 
Dealing With the Commutator Terms: $F$

We now must bound the right hand side of our energy estimates for Lie derivatives of $F$. We can write

$$J[\mathcal{L}_Z^I \tilde{F}] = [J, \mathcal{L}_Z^I] \tilde{F} + \mathcal{L}_Z^I J[F] - \mathcal{L}_Z^I J[\overline{F}].$$

We must bound these terms separately. Roughly speaking, the first term consists of error terms which behave like $\nabla_\alpha \mathcal{L}_Z^I (\nabla \cdot Z)(\mathcal{L}_Z^J F)^{\alpha \beta}$. The second term can be written in terms of $\phi$ and its derivatives. The analysis of this follows closely that in (Lindblad and Sterbenz 2006), and uses $L^2, L^\infty$ estimates along with the identity

$$\text{S} \left( \phi \overline{D_\alpha \psi} + \psi \overline{D_\alpha \phi} \right) = \text{S} \left( \frac{\phi}{r^*} \frac{D_\alpha (r^* \psi)}{r^*} + \frac{\psi}{r^*} \frac{D_\alpha (r^* \phi)}{r^*} \right),$$

along with similar symmetries.
Finally, for the last term, we use the approximation

\[\nabla_\beta F^{\alpha\beta} \approx \frac{q}{4\pi} \frac{\bar{\chi}'(r^* - t - 2)}{r^{*2}} L^*\alpha,\]

such that Lie derivatives of this quantity (and associated commutators) can be explicitly defined and bounded, and error terms depend on the deviation of the metric from Minkowski.
Dealing With the Commutator Terms: $\phi$

The energy estimates for derivatives of $\phi$ can be solved using a similar method. It suffices to show that for $|l| \leq k$ we have the estimate

$$\left\| \tau^s_+ \tau^{-1/2}_- \Box_g (D^l_X \phi) \tau^{1/2}_I \right\|_{L^2(t,x)} \lesssim \mathcal{E}_k(T).$$

first using the identity

$$[\Box^C, D_Y] \phi = -D^\alpha \phi \nabla_\alpha (\nabla \cdot Y) + D_\alpha \left( (Y) \pi^{\alpha\beta} D_\beta \phi \right) - i(\nabla^\alpha F_{Y\alpha} \phi + 2 F_{Y\alpha} D^\alpha \phi).$$

We iterate this to establish the full commutator $[\Box^C, D_Y^l] \phi$. Again, we look by this term-by-term.
Dealing With the Commutator Terms: $\phi$

The energy estimates for derivatives of $\phi$ can be solved using a similar method. It suffices to show that for $|l| \leq k$ we have the estimate

$$\left\| \tau_+^s \tau_-^{1/2} \Box_g (D^l_X \phi) \right\|_{L^2(t,x)} \lesssim \mathcal{E}_k(T).$$

first using the identity

$$[\Box_g, D_Y] \phi = -D^\alpha \phi \nabla_\alpha (\nabla \cdot Y) + D_\alpha \left( (Y)_\pi^{\alpha \beta} D^\beta \phi \right) - i(\nabla^\alpha F_{Y \alpha} \phi + 2F_{Y \alpha} D^\alpha \phi).$$

We iterate this to establish the full commutator $[\Box_g, D^l_Y] \phi$. Again, we look by this term-by-term.

The first term is again an error term which scales with the metric, and we deal with it in a similar way to the analogous term in the current norm.
Dealing With the Commutator Terms: $\phi$

The energy estimates for derivatives of $\phi$ can be solved using a similar method. It suffices to show that for $|I| \leq k$ we have the estimate

$$\left\| \tau^+_s \tau^-_1/2 \square^C g (D^I_X \phi) \right\|_{L^2(t,x)}^{1/2} \lesssim E_k(T).$$

first using the identity

$$[\square^C g, D_Y] \phi = -D^\alpha \phi \nabla_\alpha (\nabla \cdot Y) + D_\alpha \left( (Y)_\pi^{\alpha \beta} D_\beta \phi \right) - i(\nabla^\alpha F_{Y\alpha} \phi + 2 F_{Y\alpha} D^\alpha \phi).$$

We iterate this to establish the full commutator $[\square^C g, D_Y^I] \phi$. Again, we look by this term-by-term.

The first term is again an error term which scales with the metric, and we deal with it in a similar way to the analogous term in the current norm.

For the second term, $D_\alpha \left( (Y)_\pi^{\alpha \beta} D_\beta \phi \right)$, we take advantage of the fact that this is approximately equal to $c_Y \square^C g \phi$. We must be careful when bounding error quantities like

$$( (Y)_\pi^{\alpha \beta} - c_Z g^{\alpha \beta} ) D_\alpha D_\beta \phi.$$
Dealing With the Commutator Terms: $\phi$ (Cont.)

The third term requires great care even in the Minkowski metric. We first decompose

$$(\nabla^\alpha F_{\gamma\alpha})\phi + 2 F_{\gamma\alpha} D^\alpha \phi = J_{\gamma} \phi + 2 F_{\gamma\alpha} \frac{D^\alpha (r^* \phi)}{r^*} + \left( \nabla^\alpha y^\beta - 2 y^\beta \nabla^\alpha (r^*) \right) F_{\beta\alpha} \phi.$$ 

The analysis of the first two terms is straightforward, where we take advantage of the $L^2(t)L^\infty(x)$ estimate for the nice components of $F, D(r^* \phi)$ and match it with our $L^\infty(t)L^2(x)$ on our bad components. For the third term, we must establish the bound

$$\left( \nabla^\alpha y^\beta - 2 y^\beta \nabla^\alpha (r^*) \right) F_{\beta\alpha} \lesssim \alpha[F] + \rho[F] + \sigma[F] + \tau_0^{-s} \alpha[F].$$

This nice cancellation in the bad components was noted in (Shu 1991).
We can now combine our energy results. In particular, for sufficiently small $\mathcal{E}_k(0)$, it follows that

$$\mathcal{E}_k(T) \lesssim \mathcal{E}_k(0) + \epsilon_k[g]^2.$$  \hspace{1cm} (11)

The main theorem follows. We note that the initial energy is equivalent to selecting initial conditions $\dot{\phi}, D\phi_0, B_0, E_0$, respectively the initial time, and bounding certain Sobolev norms on the first three quantities as well as the divergence-free part of $E_0$. 
Thanks for listening!