This exam contains 8 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a “fundamental theorem” you must indicate this** and explain why the theorem may be applied.

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

- If you need more space, use the back of the pages; clearly indicate when you have done this.

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Do not write in the table to the right.
1. (20 points) What are the definitions of the following terms?

1. **Group.**

   **Solution:** A group is a set $G$ together with a binary operation $\cdot : G \times G \rightarrow G$ satisfying:

   (a) (associativity) For all $g_1, g_2, g_3 \in G$,
       \[ g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3. \]

   (b) (existence of an identity element) There is an element $e \in G$ such that for all $g \in G$,
       \[ g \cdot e = e = e \cdot g. \]

   (c) (existence of inverses) For each $g \in G$ there is an element $h \in G$ such that
       \[ g \cdot h = e = h \cdot g. \]

2. **Normal Subgroup.**

   **Solution:** A subset $H \subseteq G$ is called a normal subgroup of $G$ if the following conditions are satisfied:

   (a) $H$ is closed under the group law on $G$, i.e. for all $h_1, h_2 \in H$ the product $h_1 h_2 \in H$.

   (b) $H$ is a group under the restriction of the group law from $G$.

   (c) For all $g \in G$ and $h \in H$ the conjugate $ghg^{-1} \in H$. 

2. (18 points) Let $G$ and $H$ be groups. Let $f : G \to H$ be a group homomorphism. Show that $\ker(f)$ is a normal subgroup of $G$. Show that $\operatorname{Im}(f)$ is not necessarily a normal subgroup of $H$.

Solution: Let $h_1, h_2 \in \ker(f)$, then $f(h_1 h_2) = f(h_1)f(h_2) = ee = e$; thus $h_1 h_2 \in \ker(f)$. Since $f(e) = e$, we have $e \in \ker(f)$. Finally since $f(h_1^{-1}) = f(h_1)^{-1} = e^{-1} = e$, we have $h_1^{-1} \in \ker(f)$. We conclude that $\ker(f)$ is a subgroup of $G$. If $g \in G$, then

$$f(gh_1g^{-1}) = f(g)f(h_1)f(g)^{-1} = f(g)ef(g)^{-1} = f(g)f(g)^{-1} = e.$$ 

Hence, $gh_1g^{-1} \in \ker(f)$. We conclude $\ker(f)$ is a normal subgroup.

It is not necessarily the case that $\operatorname{Im}(f)$ is a normal subgroup. Consider the case where $G = \mathbb{Z}/2\mathbb{Z}$ and $H = S_3$ and $f : G \to H$ is the homomorphism mapping 1 to the 2-cycle $(1\ 2)$. The image of $f$ is not normal, as conjugation by $H = S_3$ transitively permutes the three cyclic subgroups generated by a 2-cycle.
3. (18 points) Let $G$ be a group and $H$ be a subgroup. Show that if $[G : H] = 2$, then $H$ is a normal subgroup of $G$.

Solution: Let $\varphi : G \to S_{G/H} \cong S_2$ be the homomorphism given by left multiplication on cosets. We claim $\ker(\varphi) = H$. We know that $\ker(\varphi) = \text{Core}(H) \leq H$. Conversely, observe that if $g \in G$, then the image $\varphi(g)$ is either a 2-cycle (and $g$ fixes no point of $G/H$) or $g \in \ker(\varphi)$. Since the elements $h \in H$ fix the coset $H$, it must be the case that $H \leq \ker(\varphi)$. We conclude $H = \ker(\varphi)$ is normal.
4. (45 points) Determine if the following statements are true or false. No justification is needed.

F All elements of order $n$ in $S_n$ are conjugate.

*Explanation:* The group $S_6$ contains two conjugacy classes of elements of order 6. The 6-cycles and the elements of cycle type $\{2,3\}$.

T All finite groups of order $n$ are isomorphic to a subgroup of $S_n$.

*Explanation:* $G$ acts on $G$ by left multiplication.

F Two finite groups are isomorphic if they have the same number of elements.

*Explanation:* $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ have the same order but are not isomorphic.

T For any pair of groups $G$ and $H$ there is a homomorphism $\varphi : G \to H$.

*Explanation:* Between any pair of groups $G$ and $H$ there is the trivial homomorphism defined by mapping all elements of $G$ to $e$.

F A homomorphism $\varphi : S_n \to S_m$ is either injective or has cyclic image of order at most 2.

*Explanation:* There is a surjective homomorphism from $S_4 \to S_3$. If $n \neq 4$, the claim is true.

T Every cyclic group is abelian.

*Explanation:* Powers of a generator commute.

F If $G$ and $H$ are non-trivial cyclic groups, then $G \times H$ is non-cyclic.

*Explanation:* The group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ is cyclic. It is generated by $(1,1)$.

F The automorphism group of a cyclic group is cyclic.

*Explanation:* $\text{Aut}(\mathbb{Z}/8\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

F If $G$ is a group and $\sigma, \tau \in G$. The order of the product $|\sigma \tau| \leq \text{lcm}(|\sigma|, |\tau|)$.

*Explanation:* Every element of $S_n$ is a product of elements of order 2 (2-cycles.)

F A subset $S \subseteq G$ of a finite group which is closed under the group operation is a subgroup.

*Explanation:* The set $S$ may be empty. True if $S$ is non-empty.

T If $G$ is a finite group, the order $|H|$ of a subgroup $H \leq G$ divides $|G|$.

*Explanation:* Lagrange's Theorem

F A subgroup of an infinite group has order 1 or $\infty$.

*Explanation:* The group $\mathbb{R}^\times$ contains $\{1, -1\}$. The group $GL_n(\mathbb{R})$ contains $S_n$.

F The order of a permutation $\sigma \in S_n$ divides $n$.

*Explanation:* A 2-cycle in $S_3$ is a counterexample.

F The centralizer $C_G(x)$ of an element $x \in G$ is an abelian subgroup of $G$.

*Explanation:* $C_G(e) = G$

F A subgroup $G \leq S_n$ is transitive if and only if $n$ divides $|G|$.

*Explanation:* There is a non-transitive embedding of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in $S_4$ generated by a pair of disjoint 2-cycles.
5. (24 points) Let 

\[ x := (1 \ 2)(3 \ 4)(5 \ 6). \]

1. What is the size of the conjugacy class of \( x \) in \( S_6 \)?

Solution: The conjugacy class of \( x \) in \( S_6 \) is the set of elements with cycle type \( \{2, 2, 2\} \). There are 

\[ \frac{6!}{3!2^3} = 15 \]

elements with this cycle type, as there are 6! ways to assign distinct values to \( a_1, \ldots, a_6 \) in \( (a_1 \ a_2) \ldots (a_5 \ a_6) \) from the set \{1, \ldots, 6\}, and for each assignment of \( a_1, \ldots, a_6 \) reordering the cycles (3! ways) and reordering the values contained in the cycles (2³ ways) define the same element of \( S_6 \).

2. What is the order of the centralizer of \( x \) in \( S_6 \)?

Solution: The centralizer of \( x \) is the stablizer of \( x \) under the conjugation action. Hence by the orbit stabilizer theorem:

\[ |C_{S_6}(x)| = |S_6|/|\{\text{conjugates of } x \text{ in } S_6\}| = 3!2^3 = 48. \]

3. Find an explicit element of order 3 in \( C_{S_6}(x) \).

Solution: We find an element that cyclically permutes the cycles appearing in \( x \). Recall that if \( g \in S_6 \), then

\[ gxg^{-1} = (g(1) \ g(2))(g(3) \ g(4))(g(5) \ g(6)). \]

Let \( g := (1 \ 3 \ 5)(2 \ 4 \ 6) \). Then \( gxg^{-1} = (3 \ 4)(5 \ 6)(1 \ 2) = x \) and \( g \) has order 3.
6. (25 points) 1. Let $A$ be a finite, abelian group. Show that $S_n$ contains a transitive subgroup isomorphic to $A$ if and only if $n = |A|.$

Here are two proofs. Proof 2 differs from Proof 1 as it essentially contains a proof of the orbit-stabilizer theorem. In proof 1 that work is hidden, as we appeal to the general theory of $G$-sets.

Solution 1: A transitive $A$-set $X$ is isomorphic (as an $A$-set) to $A/H$ for some subgroup $H \leq A.$ It follows that a transitive $A$-set must have size $|X| = |A/H| \leq |A|.$ The kernel of the associated map $\phi : A \to S_X$ is $Core(H),$ the largest normal subgroup contained in $H.$ When $A$ is abelian every subgroup is normal. Hence, $Core(H) = H.$ Therefore, if $A$ is abelian, then $A$ is isomorphic to its image under $\phi$ if and only if $H = \{e\},$ or equivalently if and only if $X \cong A$ (as an $A$-set). Applying this to the special case that $X = \{1, 2, 3, \ldots, n\}$ and $A \leq S_n$ acts through the restriction of the action of $S_n$ to $A,$ we obtain $A$ is isomorphic to a transitive subgroup of $S_n$ only if $n = |A|.$ Conversely, we have that if $n = |A|,$ then $A$ is isomorphic to a transitive subgroup of $S_A \cong S_n.$

Solution 2: Assume that $A \leq S_n$ is a transitive subgroup. Consider the map $ev_1 : A \to \{1, 2, \ldots, n\}$ defined by $a \mapsto a(1).$ We claim $ev_1$ is a bijection. The map $ev_1$ is surjective because $A$ is transitive. To see that $ev_1$ is injective consider elements $a, a' \in A$ such that $ev_1(a) = a(1) = a'(1) = ev_1(a'),$ or equivalently such that $a^{-1}a'(1) = 1.$ We claim $a = a',$ which is equivalent to showing $a^{-1}a'(k) = k$ for all $k \in \{1, \ldots, n\}.$ Let $x \in A$ such that $x(1) = k.$ Then

$$a^{-1}a'(k) = a^{-1}a'(x(1)) = a(x^{-1}a'(1)) = x(1) = k,$$

where the second equality holds because $A$ is abelian. We conclude $a = a'$ and $ev_1$ is a bijection. Hence, $|A| = n.$

Conversely if $|A| = n,$ then the transitive action of $A$ on $A$ by left multiplication yields an injective homomorphism $A \hookrightarrow S_A.$ Identifying $A$ with $\{1, \ldots, n\}$ as a set, gives an isomorphism $S_A \cong S_n.$ The image of $A$ under the composition $A \hookrightarrow S_A \cong S_n$ is a transitive subgroup of $S_n$ isomorphic to $A.$
2. Do there exist non-abelian groups with this property? Yes. A group $G$ will have the property that

$$G \text{ is isomorphic to a transitive subgroup of } S_n \text{ if and only if } n = |G|$$

exactly when it has the property that

for subgroups $H \leq G$, the group $Core(H) = \{e\}$ if and only if $H = \{e\}$. The group $Q_8$ has this property as $Core(H) = \langle -1 \rangle$ for any non-trivial subgroup $H \leq Q_8$, so $Q_8$ is isomorphic to a transitive subgroup of $S_n$ if and only if $n = 8$. 