Solution: Problem 3 - HW 6.

Consider a point \( x \in X \). The stabilizer of \( x \) in \( A_n \) are exactly those elements in \( S_n \) which stabilize \( x \) and are contained in \( A_n \), i.e. have sign equal to 1.
Symbolically:

\[
(A_n)_x = (S_n)_x \cap A_n = \ker (\text{sign}((S_n)_x) \to \{ \pm 1 \}).
\]

It follows

\[
| (A_n)_x | = | (S_n)_x \cap A_n | = \begin{cases}
\frac{1}{2} | S_n | & \text{if } \text{sign}((S_n)_x) = \{ \pm 1 \} \\
| S_n | & \text{if } \text{sign}((S_n)_x) = \{ 1 \}.
\end{cases}
\]

By the orbit-stabilizer theorem:

\[
|A_n \cdot X| = \frac{|A_n|}{|A_n)_x|} = \frac{\frac{1}{2} | S_n |}{| (A_n)_x |}
\]

\[
= \begin{cases}
\frac{1}{2} | S_n \cdot x | & \text{if } \text{sign}((S_n)_x) = \{ \pm 1 \} \\
| S_n \cdot x | & \text{if } \text{sign}((S_n)_x) = \{ 1 \}.
\end{cases}
\]

Since \( S_n \) acts transitively on \( X \), we have \( S_n \cdot x = X \) thus there are 2 orbits if and only if \( \text{sign}((S_n)_x) = \{ 1 \} \) (and one orbit otherwise).
It suffices to check what happens when \( x = D_{2n} \) (remember elements of \( X \) are cosets in \( S_n / D_{2n} \)).

In this case,
\[
(S_n)_x = D_{2n}.
\]

As every element of \( D_{2n} \) is a product of powers of
\[
r = (1 \ 2 \ 3 \ \ldots \ n)
\]
\[
s = (2 \ n)(3 \ n-1)\ldots\left(\frac{n+1}{2} \ \frac{n+3}{2}\right) \quad \text{(if } n \text{ is odd)}
\]
we have
\[
\text{Sign} \ (S_n)_x = \prod \beta_i \quad \text{if and only if} \quad \text{Sign} \ (r) = 1 \quad \text{and} \quad \text{Sign} \ (s) = 1.
\]

The sign of \( r \) equals 1 if and only if \( n \) is odd.
The sign of \( s \) equals 1 if and only if it contains an even number of transpositions — if \( n \) is odd this happens if and only if \( 2 | n - \frac{1}{2} \) i.e. \( n \equiv 1 \mod 4 \). \( \square \)
(c) Consider $Q_8$ acting on itself by left multiplication. This gives an injective homomorphism

$$\phi: Q_8 \rightarrow S_{Q_8}$$

Identifying $Q_8 \sim \{1, 2, 3, 4, 5, 6, 7, 8\}$ gives an isomorphism

$$S_{Q_8} \rightarrow S_8,$$

which one may pre-compose with $\phi$ to obtain an injective homomorphism

$$\gamma: Q_8 \rightarrow S_8.$$

We now do this explicitly. Consider the identification

$$
\begin{align*}
e & \mapsto 1 & j & \mapsto 5 \\
i & \mapsto 2 & ij & \mapsto 6 \\
i^2 & \mapsto 3 & -j & \mapsto 7 \\
i^3 & \mapsto 4 & ij & \mapsto 8.
\end{align*}
$$

of $Q_8$ with $\{1, 2, 3, 4, 5, 6, 7, 8\}$. 
Under left multiplication $i$ acts as follows:

\[
\begin{array}{ccc}
& i & e \\
& i & i \\
i & i & i \\
& i & i \\
& i & i \\
\end{array}
\]

\[
\begin{array}{ccc}
& j & i \\
j & -i & j \\
& -j & i \\
j & -i & j \\
j & -i & j \\
\end{array}
\]

Hence, we have (with respect to our identification)

\[6_i := \psi(i) = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8).\]

Similarly,

\[
\begin{array}{ccc}
& j & e \\
& j & j \\
i & j & j \\
& j & i \\
& j & j \\
\end{array}
\]

\[
\begin{array}{ccc}
& j & i \\
& -j & j \\
& j & i \\
& j & i \\
& j & i \\
\end{array}
\]

\[6_j := \psi(j) = (1\ 5\ 3\ 7)(2\ 8\ 4\ 6).\]
(d) Assume for the sake of contradiction that there exists $\varphi: Q_8 \rightarrow S_n$ where $n < 3$.

Then we may decompose the $Q_8$-set

$$X := \{1, \ldots, n^3\}$$

into $Q_8$-orbits, each of which have the form

$$Q_8/H$$

for some subgroup $H$ in $Q_8$. As $|X| < 8$ each of these orbits must have size less than 8, and thus the corresponding subgroups $H$ must be non-trivial.

A Note in $Q_8$ the only element of order 2 is $-1$; hence $\langle -1 \rangle$ is normal and every non-trivial subgroup of $Q_8$ contains it.

It follows $\langle -1 \rangle \trianglelefteq \text{Core}(H)$ for any non-trivial subgroup $H$.

It follows $-1$ acts trivially on $Q_8/H$ for every subgroup $H$ and thus must act trivially on $X$. 
(e) Let $n(G)$ denote the smallest $n$ such that there exists an injective homomorphism 
$$\varphi: G \longrightarrow S_n$$

By (d), $n(Q_8) = 8$.

As $8 < n!$ for $n < 4$, we have that 
$$n(G) \geq 4$$
for any group of order 8.

We've seen that $D_8$ embeds in $S_4$, and all subgroups of this order are isomorphic to $D_8$. Hence 
$$n(D_8) = 4$$

and 
$$n(G) \geq 5$$
if $G$ is a group of order 8 and $G \not\cong D_8$. 
I claim that if $|G| = 8$ then $\nu(G)$ is an even number. To see this assume that $G$ embeds in $S_X$ for some set $X$ of odd size. Then the orbits of $G$ on $X$ have order $1$, $2$, $4$ or $8$. As $|X|$ is odd, there must be an orbit of size $1$ (e.g. a point $x \in X$ stabilized by $G$). The map

$$\varphi : G \rightarrow S_X$$

must therefore have image in the stabilizer of $x \in X$, i.e.

$$\varphi : G \rightarrow S_X \rightarrow S_{X-x} \cong (S^x)_x$$

It follows $G$ embeds in a smaller symmetric group.

It follows $\nu(G) \geq 6$ if $G \not\cong D_8$.

The group $D_8$ contains $Z_4$ and $Z_2 \times Z_2$, hence if $G \cong Z_4 \times Z_2$ or $(Z_2 \times Z_2) \times Z_2$ there is an embedding

$$G \rightarrow D_8 \times Z_2 \rightarrow S_4 \times S_2 \rightarrow S_6.$$

We conclude $\nu(G) = 6$ if $G \cong Z_4 \times Z_2$ or $(Z_2)^3$. 
Finally,

\[ n(\mathbb{Z}/8\mathbb{Z}) \text{ is the smallest } n \text{ such that } S_n \text{ contains an element of order } 8. \]

The order of a permutation is the LCM of the cycle lengths appearing in its cycle decomposition. So, \( S_n \) must contain an 8-cycle and thus

\[ n(\mathbb{Z}/8\mathbb{Z}) = 8. \]
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(1) Let $G$ be a finite group and $p$ be the smallest prime dividing the order $|G|$. Let $\varphi : G \to S_p$ be a homomorphism. Show that the image $\varphi(G)$ is the trivial group $\{e\}$ or a cyclic group generated by a $p$-cycle.

The order of $|\varphi(G)|$ divides $|\text{Ker}(\varphi)|$. Hence,

$$|\varphi(G)| \mid |\text{Ker}(\varphi)|.$$

Since $\varphi(G)$ is a subgroup of $S_p$, we also have

$$|\varphi(G)| \mid |S_p| = p!.$$

It follows $|\varphi(G)|$ divides $\gcd(|G|, p!) = p$. Hence either $|\varphi(G)| = 1$ and $\varphi(G) = \{e\}$ or $\varphi(G) \cong \mathbb{Z}/p$ is the cyclic group generated by a $p$-cycle.

(2) Let $G$ be a finite group and $p$ be the smallest prime dividing the order $|G|$. Show that if $H \leq G$ is a subgroup of index $p$, then $H$ is normal in $G$.

Consider $G$ acting on $G/H$ by left multiplication. This yields a homomorphism $\Phi : G \to S_{G/H} \cong S_p$. The image of $H$ is a subgroup of the image of $G$, hence is either trivial or equals the cyclic subgroup generated by a $p$-cycle of the form $(H, g_1H, g_2H, \ldots, g_pH)$.

But under left multiplication every element of $H$ fixes the coset $H$, hence the image of $H$ under $\Phi$ is trivial. Thus, $\text{Ker}(\Phi) = H$. We conclude that $H$ is a normal subgroup (as kernels are normal).