(1)(a) \( \leq \) Without loss of generality, we may assume \( L_i = g_iH_i \).

It suffices to show \( \Theta_{L_i} = \Theta_{g_iH_i} \).

Define \( \varphi: \Theta_{L_i} \to \Theta_{g_iH_i} \)

\[ gH_i \mapsto g\varphi(g_iH_i) = gH_i \]

\( \varphi \) is well-defined and injective, since \( gH_i = g'H_i \iff g' \in H_i \iff gH_i = g'H_i \).

and \( \varphi \) is \( \Theta \)-homomorphism: \( \varphi(g'H_i) = g\varphi(g'H_i) = gH_i \).

\( |\Theta_{L_i}| = |\Theta_{g_iH_i}| \), hence injectivity implies surjectivity, \( \varphi \) is an isomorphism.

\( \Rightarrow \): if \( \varphi: X_i \to X_0 \) is isomorphic, it maps orbits to orbits, in particular they have the same number of orbits, \( n = n' \), and the orbits are isomorphic accordingly. In other words, \( \varphi \) induces \( C \in S_n \) from \( X_i \) orbits to \( X_0 \) orbits, and the restriction of \( \varphi \) on \( \Theta_{L_i} \) is isomorphism (\( \rho: \Theta_{L_i} \to \Theta_{L_0} \)).

It remains to show \( H_i \) and \( L_0 \) are conjugate.

\( \forall H, K \leq G \), if \( \varphi: \Theta_{L_i} \cong \Theta_{L_0} \) as \( \Theta \)-sets, then \( \varphi \) is determined by \( \varphi(H) \):

suppose \( \varphi(H) = g_0K \) for some \( g_0 \in G \), then \( \varphi(\gamma H) = g\varphi(\gamma H) = g\gamma g_0K \).

Since \( \varphi \) is well-defined and injective, \( \gamma H = \gamma g_0K \iff \gamma g_0K \gamma^{-1} = g_0 \).

\( \Rightarrow K = g_0^{-1}Hg_0 \).

(b) \( g \in G \) acts trivially on \( \Theta_{L_i} \) iff \( g \in \text{Core}(H_i) \) as shown before.

\( \ker \varphi = \{ g \in G : g \text{ acts trivially on every } \Theta_{L_i} \} = \cap \text{Core}(H_i) \).

(c) next page

(d) From (a), conjugacy classes of subgroups of \( S_6 \) isomorphic to \( S_4 \) are bijective to isomorphism types of faithful \( S_4 \)-sets of size 6. It follows from (c) that there are 4.

(4) In cases (2) and (3), \( \varphi(S_4) \) in \( S_6 \) is uniquely defined by a choice of 4 elements, where there are \( \binom{6}{4} = 15 \) options. So each class has 15 subgroups.

In cases (3) and (4), the 6 orbits come in 3 pairs, where each pair have the same stabilizer.

\[ S_4 = \{ H, nH, gH, gnH, gH, gnH \}. \]

\( n \in N_{S_4}(H) \), \( H \cap \langle w \rangle = \{ 1, w \} \).

Under \( S_4 \)-action, pairs are mapped to pairs, since \( gg_i H = g g_i H \).

Therefore after fixing a partition of 6 elements into 3 pairs, say \( \{ 1, 2, 3 \} \), \( \{ 4, 5, 6 \} \).
(2) **Isomorphism Type:** $S_4 \times S_4$

**Kernel:** Faithful
**Transitive:** Alternating
**$V$-Set Type:** $V$
**Reason For $V$-Set:** (ab) $\rightarrow$ (12)

<table>
<thead>
<tr>
<th>$S_4$</th>
<th>$N$</th>
<th>$N$</th>
<th>$V$</th>
<th>$V$</th>
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</thead>
<tbody>
<tr>
<td>$S_4$</td>
<td>$N$</td>
<td>$N$</td>
<td>$V$</td>
<td>$V$</td>
</tr>
</tbody>
</table>

(3) $S_4 \langle (123) \rangle$

**$V$-Set Type:** $V$
**Reason For $V$-Set:** $\langle (123) \rangle$ $\rightarrow$ (2,2)-cycle.

(4) $S_4 \langle (12) \rangle$

**$V$-Set Type:** $V$
**Reason For $V$-Set:** $\langle (12) \rangle$ $\rightarrow$ (2,2)-cycle.

(3) **Bonn**

(i) $S_4$ sits inside the subgroup $K = \{ \sigma \in S_6 : \sigma(1) = \sigma(2), \sigma(3) = \sigma(4) \}$

$|K| = S_3 \times S_2 = 2 |S_4|$. $S_4$ index 2 in $K \Rightarrow S_4 \triangleleft K$.

and $N_{S_6}(S_4)$ cannot be any larger, in order to preserve pairs, thus $N_{S_6}(S_4) = K$, the number of $S_4$ conjugates $= \frac{|S_6|}{|K|} = 15$.

(ii) $X = X^K \cup \left( \bigcup_{i} \text{orb}(x_i) \right)$ $x_i$'s are representatives of orbits with more than one element.

$|\text{orb}(x_i)| = \frac{|H|}{|\text{stab}(x_i)|}$. $|\text{orb}(x_i)| |H|$ and $|H| > 1 \Rightarrow |\text{orb}(x_i)| \equiv 0 \mod p$. $\Rightarrow |X| = |X^K| + \left( \bigcup_{i} |\text{orb}(x_i)| \right)$.

(b) $H = \langle o \rangle$. Let $o$ act on $X$ by $(g_0, g_1, \ldots, g_{p-1}) \rightarrow (g_0, \ldots, g_{p-1}, g_0) \in X, g_0 = (g_0, g_2)$

This permutation on $X$ has order $p$, thus defines an $H$-action on $X$.

(c) $\pi^H : G^H \rightarrow X$

$(g_0, g_1, \ldots, g_{p-1}) \rightarrow (g_0, \ldots, g_{p-1}, g_0)$ is the inverse map.

(d) $X^H = \{(g_0, \ldots, g_{p-1}) \in G^p : g_0 \cdots g_{p-1} = e, (g_0, \ldots, g_{p-1}, g_0) = (g_0, \ldots, g_{p-1}) \}$

$= \{(g_0, \ldots, g_{p-1}) \in G^p : g_0 = e \} = \{(g_0, \ldots, g_{p-1}) \in G^p : g_0 = e \} \rightarrow g_0$ is a bijection.
(e) \(|H| = |x| = |G|^{p-1} = 0 \pmod{p}\). And it's not 0 since \(e\) has order dividing \(p\).

\[ 3) (a) \quad \forall g_1, g_2, x, \quad (g_1g_2)(x) = g_1(g_2xg_2^{-1}) = g_1(g_2x)g_1^{-1} = g_1 \cdot g_2(x). \]

I.e., \(g_1g_2 = g_1 \cdot g_2\).

(b) (i) By Cauchy's Thm, \(G\) has a subgroup \(K\) of order \(p\).

The left multiplication on left \(K\)-cosets defines \(G \rightarrow S_g K\).

\(\ker \phi = \text{Core}(K) \leq K\). on the other hand \(|S_g K| = k!\) not divisible by \(p\).

It has to be the case \(\ker \phi = K\). \(\Rightarrow\) \(K\) normal. \((G: K) = \text{Aut}(K)\) exists.

\(K\) is abelian as itself, so \(K \leq \text{ker} \phi\). \[\text{Im} \phi = \frac{|G|}{|\ker \phi|} = \frac{|G|}{|K|} = k.\]

(ii) \(|\text{Aut}(K)| = p-1. \quad \text{Im}(\psi_K) \geq 1\) by (i).

But \(q_2 > p-1\). \(\text{Im}(\psi K)\) can only be \(1\), in which case \(\ker \psi K = G\).

Take \(x, y \in G\) of order \(p, q\) respectively, then \(x, y\) commute. \(o(xy) = pq\).

\(G\) is cyclic, in particular abelian.

This proves Theorem 0.2.

(c) A 3-cycle and 5-cycle commute in \(S_n\) iff they are disjoint.

If \(n < 8\), there are no such elements. By (b) \(S_n\) doesn't contain order 15 subgroup.

(d) If \(H \leq S_5\) transitive, \(|H| = 5! / \text{Stab}(i)\) divisible by \(5\). And thus contains 5-cycles by Cauchy.

\(|H|\) may be \(120, 60, 40, 30, 20, 15, 10, 5\).

40 and 30 are excluded by discussion in class.

120: \(H = S_5\).

60: \(H = A_5\).

Otherwise \(|H| \leq 20\), then order 5 subgroups in \(H\) are normal by (b)(i).

I.e. \(H\) is a subgroup of the normalizer of a 5-cycle with order divisible by \(5\).

\[|N_{S_5}(\langle c \rangle)| = \frac{|S_5|}{|\langle c \rangle|} = 6 \quad \text{b/c all 5-cycles are conjugate}.\]

There are \(\frac{4!}{4} = 6\) order 5 subgroups. \(\Rightarrow\) \(|N_{S_5}(\langle c \rangle)| = \frac{120}{6} = 20\).
(e) by (i), K is normal, we have \( \varphi_k : G \to \text{Aut}(K) \) and \( \text{Im}\varphi_k \leq 2 \).

If \( \text{Im}\varphi_k = \{1\} \), then \( \ker \varphi_k = G \).

Take \( x \) of order 2, \( y \) a generator of \( K \). then \( x, y \) commute. \( G \cong \langle x, x, y \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \).

If \( |\text{Im}\varphi_k| = 2 \), it contains an order 2 element in \( \text{Aut}(K) \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \).

If \( [n] \in (\mathbb{Z}/2)^x \) has order 2, \( m \equiv 1 \pmod{p} \),

\[(n-1)(m+1) \equiv 0 \pmod{p}, \text{ but } m \neq 0, \text{ it has to be } nm + 1 \equiv 0.

Thus we've shown the only order 2 element is \( [-1] \). \( \text{Im}\varphi_k = \{ [1], [-1] \} \).

Now take a \( s \in G \) s.t. \( \varphi_k(s) = [-1] \), \( r \) a generator of \( K \).

Then \( srs^{-1} = \varphi_k(s)(r) = r^{-1} \).

\( \varphi_k(s^2) = \varphi_k(s)^2 = [1] \). \( s^2 \in \ker \varphi_k = K \).

If \( s^2 \neq e \), this means \( s \) has order \( 2p \), contradictory to \( |\text{Im}\varphi_k| = 2 \).

So \( s^2 = e \). \( r \) and \( s \) satisfy the relations of \( D_{2p} \). \( G \cong D_{2p} \).

This proves Theorem 0.3.